MATH 211 Basic Algebra

Problem Set 4

1. Prove that if A, B are normal subgroups of G, and $A \cap B = \{e\}$ then $ab = ba$ for any $a \in A$, $b \in B$.
2. Find all normal subgroups in the groups S_3, S_4, A_4.
3. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
4. Find all homomorphisms $\mathbb{Z}_6 \to \mathbb{Z}_6$, $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$, $\mathbb{Z}_6 \to \mathbb{Z}_{18}$, $\mathbb{Z}_{18} \to \mathbb{Z}_6$, $\mathbb{Z}_{12} \to \mathbb{Z}_{15}$, and $\mathbb{Z}_6 \to \mathbb{Z}_{25}$.
5. Find all automorphisms of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{12}$.
6. Prove that there is no epimorphism of $\mathbb{Q}$ onto $\mathbb{Z}$
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8. Prove that $\mathbb{R}/\mathbb{Z} \simeq U$, $\mathbb{C}^*/\mathbb{R}^* \simeq U$, $U/\mathbb{C}(n) \simeq U$, $\mathbb{C}^*/\mathbb{C}(n) \simeq U$, $\mathbb{C}(p^\infty)/\mathbb{C}(p^n) \simeq \mathbb{C}(p^\infty)$.
9. Find quotient-groups: $4\mathbb{Z}/12\mathbb{Z}$, $n\mathbb{Z}/nm\mathbb{Z}$, $\mathbb{C}(15)/\mathbb{C}(5)$, $\mathbb{C}(nm)/\mathbb{C}(n)$.
10. The set $Z(G) = \{g \in G : \forall x(xg = gx)\}$ is called the center of the group G. Prove that $Z(G)$ is invariant under any automorphism of G, and in particular is a normal subgroup of G.
11. Prove that $\mathrm{Inn}(G) \simeq G/Z(G)$.