Ordinal Arithmetic

Summer Midterm III
18th of June, 1999
Ali Nesin

Let α and β be two ordinals. As in the previous midterm, well-order the set $(\alpha \times \{0\}) \cup (\beta \times \{1\})$ by putting the elements of β to the end of α. As every set, this new well-ordered set is isomorphic to a unique ordinal, that we will call $\alpha + \beta$.

Below, α, β, γ denote arbitrary ordinals.

1. Show that $0 + \alpha = \alpha + 0 = \alpha$ and $\alpha + 1 = \alpha^+$

2. Show that $n + \alpha = \alpha$ if $\alpha > \omega$ and $n \in \omega$.

3. Show that if $\beta < \alpha$, then $\beta + \gamma = \alpha$ for some γ.

4. Show that if $\alpha + \beta = \alpha + \gamma$ for some α, then $\beta = \gamma$.

5. Say, without necessarily a rigourous proof, why

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

Let α and β be two ordinals. Order the set $\alpha \times \beta$ as follows:

$$(a, b) < (a', b') \text{ iff either } b < b' \text{ or } (b = b' \text{ and } a < a').$$

This is the reverse lexicographic order. As we have seen in the previous midterm, it is a well-order and therefore is isomorphic to a unique ordinal $\alpha\beta$.

Below, α, β, γ denote arbitrary ordinals.

6. Show that $\alpha 0 = 0\alpha = 0$ and $1\alpha = \alpha 1 = \alpha$.

7. Show that $2\omega = \omega$.

8. Show that $\omega\omega \neq \omega$.

8. One can show that the equalities $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ hold. Show that the equality $(\alpha + \beta)\gamma = \alpha\beta + \beta\gamma$ does not hold in general.

9. For ordinals α, β and λ define

$$\alpha^0 = 1$$
$$\alpha^{\beta+1} = \alpha^\beta \alpha$$
$$\alpha^\lambda = \bigcup_{\beta<\lambda} \alpha^\beta \text{ if } \lambda \text{ is a limit ordinal.}$$

Show that $0^\alpha = 0$ and $1^\alpha = 1$.

10. Show that $2^\omega = \omega$.

11. One can check that the equalities $\alpha^{\beta + \gamma} = \alpha^\beta \alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$ always hold. Show that the equality $(\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$ does not always hold.