Math 111
Midterm 2
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**Hint:** Do not be frightened by the look of the questions.

**I. Combinatorics.**
Given a natural number $n$, define $n!$ by induction as follows.

$$0! = 1$$

$$(n + 1)! = n!(n + 1)$$

For natural numbers $0 \leq k \leq n$, define,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**I.1.** Show that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

**I.2.** Conclude that $\binom{n}{k}$ is a natural number.

**I.3.** Show that for all natural numbers $n$ and rational numbers $x$ and $y$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**II. Logarithmic Constant $e$**

**II.1.** Show that an increasing sequence which is bounded above is a Cauchy sequence.

**II.2.** Show that $n! > 2^n$ for large enough $n$.

**II.3.** Conclude from **I.3** and **II.2** that the sequence $a_n = (1+1/n)^n$ is bounded above (e.g. by $2 + 19/24$).

**II.4.** Show that $a_n = (1+1/n)^n$ is an increasing sequence for $n > 1$. (**Hint:** Show first that $(x-1)^n \geq x^n - nx^{n-1}$ for $x \geq 1$. Then show that $a_{n-1} \leq a_n$).

**II.5.** Conclude that the sequence $a_n = (1+1/n)^n$ is a Cauchy sequence.

**III. Finiteness**

We give four definitions of finiteness.

We call a set $X$ **1-finite** if there is a bijection between $X$ and a natural number.

We call a set $X$ **2-finite** if every nonempty set of subsets of $X$ has a maximal element (for inclusion), i.e. if $\forall Y \subseteq P(X) \exists A \in Y \forall B \in Y (A \subseteq B \rightarrow A = B)$.

We call a set $X$ **3-finite** if every nonempty set of subsets of $X$ has a minimal element (for inclusion).

We call a set $X$ **4-finite** if there is no bijection between $X$ and a proper subset of $X$.

**III.1.** Show that a set is 2-finite if and only if it is 3-finite.

**III.2.** Show that a natural number is 2-finite.

**III.3.** Conclude that if a set is 1-finite then it is 2 and 3-finite.

**III.4.** Show that if a set is 2 or 3-finite then it is 4-finite.

**III.5.** Conclude that the set $\mathbb{N}$ of natural numbers is not $i$-finite for any $i = 1,2,3,4$.

**III.6.** Conclude that every natural number is $i$-finite for any $i = 1,2,3,4$. 