Math 111

Midterm 2

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Hint: Do not be frightened by the look of the questions.

I. Combinatorics.

Given a natural number n, define $n!$ by induction as follows.

$$0! = 1$$
$$(n + 1)! = n!(n + 1)$$

For natural numbers $0 \leq k \leq n$, define, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

I.1. Show that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

I.2. Conclude that $\binom{n}{k}$ is a natural number.

I.3. Show that for all natural numbers n and rational numbers x and y,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

II. Logarithmic Constant *e*

II.1. Show that an increasing sequence which is bounded above is a Cauchy sequence.

II.2. Show that $n! > 2^n$ for large enough n.

II.3. Conclude from **I.3** and **II.2** that the sequence $a_n = (1+1/n)^n$ is bounded above (e.g. by 2 + 19/24).

II.4. Show that $a_n = (1+1/n)^n$ is an increasing sequence for $n > 1$. (**Hint:** Show first that $(x - 1)^n \geq x^n - nx^{n-1}$ for $x \geq 1$. Then show that $a_{n-1} \leq a_n$).

II.5. Conclude that the sequence $a_n = (1+1/n)^n$ is a Cauchy sequence.

III. Finiteness

We give four definitions of finiteness.

We call a set X **1–finite** if there is a bijection between X and a natural number.

We call a set X **2–finite** if every nonempty set of subsets of X has a maximal element (for inclusion), i.e. if $\forall Y \subseteq \boldsymbol{P}(X)\ \exists A \in Y\ \forall B \in Y\ (A \subseteq B \rightarrow A = B)$.

We call a set X **3–finite** if every nonempty set of subsets of X has a minimal element (for inclusion).

We call a set X **4–finite** if there is no bijection between X and a proper subset of X.

III.1. Show that a set is 2-finite if and only if it is 3-finite.

III.2. Show that a natural number is 2-finite.

III.3. Conclude that if a set is 1-finite then it is 2 and 3-finite.

III.4. Show that if a set is 2 or 3-finite then it is 4-finite.

III.5. Conclude that the set $\mathbb{N}$ of natural numbers is not i-finite for any i = 1,2,3,4.

III.6. Conclude that every natural number is i-finite for any i = 1,2,3,4.