Math 111

Third Midterm (Ordinals)
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To solve a question, you may assume you have solved all the previous questions. Take your time. Start from the first question and go in that order. Only rarely, and certainly after giving a long thought, that you should skip a question that you were unable to solve. Do not try to cheat, you won't succeed. Do not lie to yourself, be honest. Do not try to succeed at any rate. Write clearly, with brieve and correct English sentences. Do not use symbols like $\forall$, $\exists$, $\Rightarrow$ unnecessarily.

Let $(X, <)$ be a totally ordered set. We say that $(X, <)$ is a **well-ordered** set (or that $<$ well-orders X) if every nonempty subset of X contains a minimal element (a least) for that order, i.e., if for every nonempty subset A of X, there is an $m \in A$ such that $m \leq a$ for all a in A. Clearly, given A, such an m is unique. In particular if $X \neq \emptyset$, X has a least element.

1. Give two examples of well-ordered sets one of which is infinite. (2 pts.)

2. Show that the minimal element m of a nonempty subset A of a totally ordered set $(X, <)$ is unique. (3 pts.)

If X is a set, we set $X^+ = X \cup \{X\}$.

2. Assume X is a well-ordered set and that $X \notin X$. Order X^+ by extending the order of X and by stating that X is larger than its elements (i.e. to get X^+, put the element X to the very end of X). Show that X^+ is also a well-ordered set. (5 pts.)

3. If $(X, <)$ is an ordered set and $x \in X$, we define the **initial segment** of x as

$$s(x) = \{y \in X : y < x\}.$$

What is $s(o)$ where o is the minimal element of X? (2 pts.)

4. Let X and Y be two well-ordered sets. Let

$$A = (X \times \{0\}) \cup (Y \times \{1\}).$$

Order A as follows: For all $x, x_1, x_2 \in X$ and for all $y, y_1, y_2 \in Y$

$(x_1, 0) < (x_2, 0)$ iff $x_1 < x_2$.
$(y_1, 1) < (y_2, 1)$ iff $y_1 < y_2$.
$(x, 0) < (y, 1)$.

Show that the above relation well-orders A. (4 pts.)

5. (Transfinite Induction) Let $(X, <)$ be a well-ordered set and let $A \subseteq X$ be such that for all $x \in X$, if $s(x) \subseteq A$, then $x \in A$. Show that $A = X$. (10 pts.)

An **ordinal** is a well-ordered set α such that $\beta = s(\beta)$ for all $\beta \in \alpha$. Thus an ordinal is a set α well-ordered by the relation $\in$, i.e. the binary relation $<$ on α defined by "$\beta < \gamma$ iff $\beta \in \gamma$" well-orders α.

6. Show that $\varnothing$ is an ordinal. (2 pts.)

7. Show that if $\alpha \neq \varnothing$ is an ordinal, then $\varnothing \in \alpha$ and $\varnothing$ is the least element of α. (7 pts.)

8. Show that if α is an ordinal and $\beta \in \alpha$, then $\beta \subset \alpha$. (2 pts.)

9. Show that every element of an ordinal is an ordinal. (2 pts.)

10. Show that if α is an ordinal, then α^+ is also an ordinal. (2 pts.)

9. Let α be an ordinal and $\beta \in \alpha$. Show that either $\beta^+ \in \alpha$ or $\beta^+ = \alpha$. (8 pts.)

10. In exercise 3 take $X = \omega$ and $Y = 1 = \{0\}$. Show that the well-ordered set A obtained there is isomorphic to the ordinal ω^+, i.e. there is an order-preserving bijection from A onto ω^+. (4 pts.)

11. In exercise 3 take $X = 1 = \{0\}$ and $Y = \omega$. Show that the well-ordered set A obtained there is **isomorphic** to ω, i.e. there is an order-preserving bijection from A onto ω. (4 pts.)

12. Let α, β be ordinals. Show that either $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$. (18 pts.)

13. Show that the union of a set of ordinals is an ordinal. (3 pts.)

14. Let α and β be two ordinals. Let $f : \alpha \to \beta$ be a strictly increasing function. Show that if f is onto, then $\alpha = \beta$ and f is the identity map. (18 pts.)

15. Show that every well-ordered set is isomorphic to an ordinal. (18 pts.)