Math 111

Midterm
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Recall that $A \Delta B = (A \setminus B) \cup (B \setminus A)$, $P(X)$ = The set of subsets of X, $\mathbf{N} = \{0, 1, 2, 3, ...\}$ is the set of natural numbers (we assume it is a set).

I. Let X be a set. Show that there is no function from X onto $P(X)$.

II. We define a total order on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(x, y) < (z, t) \text{ iff either } x < z \text{ or } (x = z \text{ and } y < t).$$

IIa. Find the elements of the set $\{\alpha \in \mathbb{N} \times \mathbb{N} : (2, 3) < \alpha < (3, 2)\}$.

IIb. Can $(\mathbb{N} \times \mathbb{N}, <)$ be isomorphic to $(\mathbb{N}, <)$? Justify your answer.

III. An **equivalence relation** on a set X is a subset E of $X \times X$ such that

i) $(x, x) \in E$ for all $x \in X$.

ii) If $(x, y) \in E$ then $(y, x) \in E$ for all $x, y \in X$.

iii) If $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$ for all $x, y, z \in X$.

Very often, one writes $x \equiv y$ instead of $(x, y) \in E$.

For example, $E = \{(x, x): x \in X\}$ is an equivalence relation on X.

Let Y be a set and set $X = P(Y)$ (the set of subsets of X). On X, define,

$$A \equiv B \text{ iff } A \Delta B \text{ is finite.}$$

IIIa. Show that this defines an equivalence relation on X, i.e. show that

i) $A \equiv A$,

ii) If $A \equiv B$ then $B \equiv A$,

iii) If $A \equiv B$ and $B \equiv C$ then $A \equiv C$.

IIIb. Find all $A \in X$ such that $A \equiv \emptyset$.

IIIc. Find all $A \in X$ such that $A \equiv$ X.

IV. (Cantor-Schröder-Bernstein) Let A be a set and A' a subset of A. Assume that there is a bijection $f: A \to A'$ betweeen A and A'. Let $A' \subseteq B \subseteq A$. The purpose of this exercise is to show that there is a bijection between B and A.

Let $Q = B \setminus A'$.

Let $\Gamma = \{X \subseteq A : Q \cup f(X) \subseteq X\}$. Note that

Let $T = \cap \Gamma = \bigcap_{X \in \Gamma} X$.

IVa. Show that $T \in \Gamma$.

IVb. Show that $Q \cup f(T) \in \Gamma$.

IVc. Show that $T = Q \cup f(T)$. (Hint: Use a and b).

IVd. Show that $B = T \cup (A' \setminus f(T))$. (Hint: Use c).

IVe. Show that $T \cap (A' \setminus f(T)) = \emptyset$.

IVf. Show that there is a bijection between B and A. (Hint: Use d and e).

V. Let A and B be two sets. Assume that there are one-to-one maps f: A $\to B$ and $g: B \to A$. Show that there is a bijection between A and B. (Hint: Consider $gf(A)$ and use problem IV).