Math 211 (Algebra)

Resit
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Rewiew of some of the definitions: G always denotes a group. For an element a of G, we define the **conjugacy class** of a to be the set

$$a^G = \{g^{-1}ag : g \in G\}.$$

We define the **centralizer** of a to be

$$C_G(a) = \{g \in G : ga = ag\}.$$

The **center** of G is the set

$$Z(G) = \{z \in G : zg = gz\}.$$

If H is a subgroup of G, the **normalizer** of H is

$$N_G(H) = \{g \in G \text{ such that } gH = Hg\},$$

the **left coset space** of H is

$$G/H = \{gH : g \in G\},$$

and for $a \in G$,

$$H^a = \{a^{-1}ha : h \in H\}.$$

1. Show that

1a. $Z(G) \lhd G$. (2 pts.)

1b. $C_G(a) \leq G$. (2 pts.)

1c. $N_G(H) \leq G$. (2 pts.)

1d. $H \lhd N_G(H)$. (2 pts.)

1e. $N_G(H)$ is the largest subgroup of G that contains H and in which H is normal. (4 pts.)

2a. Find all the centralizers in Sym(4). (3 pts.)

2b. Find all subgroups of Sym(4). (3 pts.)

2c. Find all the normal subgroups of Sym(4)? (3 pts.)

2d. Find all the conjugacy classes in Sym(4). (3 pts.)

3a. Show that any two conjugacy classes in a group are either equal or disjoint. (4 pts.)

3b. Show that the conjugacy classes partition G. (2 pts.)

3c. Show that the map $gC_G(a) \to gag^{-1}$ defines a bijection between $G/C_G(a)$ and a^G. (4 pts.)

3d. Show that $a \in Z(G)$ iff $|a^G| = 1$ iff $C_G(a) = G$. (2 pts.)

3e. Assume G is finite. Show that $|G| = |Z(G)| + \sum_{\text{certain } a \notin Z(G)} |G/C_G(a)|$. (4 pts.)

3f. Assume G is a finite group of order p^n for some prime number p and a natural number n. Show that $Z(G) \neq 1$. (4 pts.)

4. Proceeding as in question 3c show that there is a bijection between $G/N_G(H)$ and the set $\{H^g : g \in G\}$. (6 pts.)

5. Let $\varphi : \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be given by $\varphi(x, y, z) = (2x - 3y, 4x - 3z)$.
5a. Show that φ is a homomorphism of groups. (2 pts.)
5b. Is φ onto? (2 pts.)
5c. Show that $\text{Ker}(\varphi) = \{(3a, 2a, 4a) : a \in \mathbb{Z}\}$. (2 pts.)

6. Show that $(\mathbb{Q} \times \mathbb{Q}) / \delta(\mathbb{Z} \times \mathbb{Z}) \approx \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}$ where

$$\delta(\mathbb{Z} \times \mathbb{Z}) = \{(z, z) : z \in \mathbb{Z}\}.$$

(10 pts.)

7a. Find a generator of the subgroup of $\mathbb{Q}^*$ generated by 2/5 and 4/7. (2 pts.)
7b. Is the subgroup of $\mathbb{Q}^+$ generated by $\{2^n : n \in \mathbb{Z}\}$ cyclic? (5 pts.)
7c. Is the subgroup of $\mathbb{Q}^*$ generated by $\{2^n : n \in \mathbb{Z}\}$ cyclic? (3 pts.)
7d. Show the subgroup of $\mathbb{R}^*$ generated by $\sqrt{2}$ and $\sqrt{3}$ is isomorphic to the subgroup of $\mathbb{R}^+$ generated by $\sqrt{2}$ and $\sqrt{3}$. (4 pts.)

8a. Find three elements of order 2 of $\mathbb{R}^*/\langle\sqrt{2}\rangle$. (3 pts.)
8b. Is there an element of infinite order in $\mathbb{Q}/\mathbb{Z}$? (3 pts.)
8c. Is there an element of finite order in $\mathbb{R}/\mathbb{Q}$? (3 pts.)

9. Find a proper nontrivial normal subgroup of $\text{Sym}(\mathbb{N})$. (6 pts.)

10. Show that $\text{Sym}(\mathbb{N}) \approx \text{Sym}(\mathbb{Z})$. (5 pts.)

11. Find a nonabelian group of order 8. (5 pts.)