Introduction to Set Theory and Number Systems
Remake Exam
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Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote respectively the set of natural numbers, integers, rational numbers and real numbers.

Note: Questions may or may not be independent of the preceeding ones. You should attempt to solve all the questions.

1. Let n be a fixed natural number. On the set $\mathbb{N}$ of natural numbers, define the relation,

$$x \equiv y \Leftrightarrow x + y \text{ is divisible by } n.$$

For what values of n is this an equivalence relation? For each such n, find $\mathbb{N}/\equiv$.

2. Let $f\colon X \to Y$ be a function. Recall that if A is a subset of Y, $f^{-1}(A)$ is defined to be as the set $\{x \in X\colon f(x) \in A\}$. If A is a subsets of Y, what is the relationship (in terms of inclusion and equality) between the sets $f^{-1}(A^c)$ and $f^{-1}(A)^c$. (Note that A^c denotes the set $Y \setminus A$.)

3. If X and Y are two topological spaces, a map $f\colon X \to Y$ is called **continuous**, if for any open subset U of Y, $f^{-1}(U)$ is an open subset of X. Show that a map $f\colon X \to Y$ is continuous if for any closed subset C of Y, $f^{-1}(C)$ is a closed subset of X.

4. Let X be any set. Show that there is no bijection between X and the set $\wp(X)$ of subsets of X. (**Hint:** Assume there is such a bijection, call it f:

$$f\colon X \longrightarrow \wp(X).$$

Let $A = \{x \in X\colon x \notin f(x)\}$. Let $a \in X$ be such that $f(a) = A$. Try to decide whether $a \in A$). Did you, by any chance, show more than what I have asked?

5. Show that if $f \circ g$ (composition of two functions) is one-to-one, then g is also one-to-one. Give an example where $f \circ g$ is one-to-one, but f is not. State and prove a similar result where "one-to-one" is replaced bu "onto".

6. Show that the rule $f(x) = x^2 - x$ defines a map from $\mathbb{N}$ into $\mathbb{N}$. Is this a one-to-one map? Is it onto? The same question with $\mathbb{N} \setminus \{0,1\}$ instead of $\mathbb{N}$.

7. Find all bijections $f\colon \mathbb{N} \to \mathbb{N}$ such that if $x < y$ then $f(x) < f(y)$. The same question with $\mathbb{Z}$ instead of $\mathbb{N}$.