Introduction to Set Theory and Number Systems
Final Exam
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Ali Nesin
Bilgi University

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote respectively the set of natural numbers, integers, rational numbers and real numbers.

Note: Questions may or may not be independent of the preceeding ones. You should attempt to solve all the questions.

1a. By using the axioms of set theory show that a function (as it is defined during the lectures) is a set.

1b. Show that the collection of functions from a set X into a set Y is a set.

1c. Let f: $X \longrightarrow Y$ be a function. Recall that if A is a subset of Y, $f^{-1}(A)$ is defined to be as the set $\{x \in X: f(x) \in A\}$. If A and B are two subsets of Y, what is the relationship (in terms of inclusion and equality) between the sets

$$f^{-1}(A) \cup f^{-1}(B),\ f^{-1}(A \cup B), f^{-1}(A) \cap f^{-1}(B) \text{ and } f^{-1}(A \cap B)?$$

1d. Show that the same kind of relations hold for an infinite family $(A_i)_{i \in I}$ of subsets of Y (instead of just two subsets).

1e. Let f: $X \to Y$ be a function and A a subset of Y. Is $A = f(f^{-1}(A))$ always? If so prove it, otherwise give a counterexample.

1f. Let f: $X \to Y$ be a function and A a subset of X. Is $A = f^{-1}(f(A))$ always? If so prove it, otherwise give a counterexample.

2a. Show that if $f \circ g$ (composition of two functions) is one-to-one, then g is one-to-one. Give an example where $f \circ g$ is one-to-one, but f is not.

2b. State a statement similar to the one above with "onto" instead of "one-to-one".

2c. Let f: $X \longrightarrow X$ be a function. Show that if f^n is a bijection for some positive integer n, then f is also a bijection. (Recall that f^n is f composed with itself n times).

3. If X and Y are two topological spaces, a map f: $X \to Y$ is called **continuous**, if for any open subset U of Y, $f^{-1}(U)$ is an open subset of X.

3a. Let X be a topological space and f: $X \to \mathbb{R}$ a function. Show that if the inverse image $f^{-1}(a,b)$ of an open interval (a,b) of $\mathbb{R}$ is open in X, then f is continuous. (R is endowed with the usual topology generated by the open intervals).

3b. By using 3a, show that the function f: $\mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous.

3c. Give an example of a noncontinuous function from $\mathbb{R}$ into $\mathbb{R}$.

3d. Assume the only open subsets of X are $\varnothing$ and X. How can one characterize the continuous functions from X into a topological space Y?

3e. Find an infinite topological space X such that every function f from X into any topological space Y is continuous.

3f. As $\mathbb{Z}$ is a subset of $\mathbb{R}$ and $\mathbb{R}$ is a topological space (in the usual sense), we may endow $\mathbb{Z}$ with the induced topology. What are the open subsets of $\mathbb{Z}$ with this topology?

4a. Let x be any set. Is there always a bijection between the sets $\mathbb{N}$ and $\mathbb{N} \cup \{x\}$?

4b. Find a bijection between $\mathbb{N}$ and $\mathbb{Q}$. If f is this bijection, what is $f(10)$?

4c. Is there a bijection f between $\mathbb{N}$ and $\mathbb{Q}$ that preserves the order, i.e. for which $x < y$ implies $f(x) < f(y)$?

5. Let X be any set. Show that there is no bijection between X and the set $P(X)$ of subsets of X. (**Hint:** Assume there is such a bijection, call it f:

$$f: X \longrightarrow \wp(X).$$

Let $A = \{x \in X: x \notin f(x)\}$. Let $a \in X$ be such that $f(a) = A$. Try to decide whether $a \in A$). Did you, by any chance, show more than what I have asked?