Introduction to Set Theory and Number Systems
Final Exam
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Notation: \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Q} \) and \( \mathbb{R} \) denote respectively the set of natural numbers, integers, rational numbers and real numbers.

Note: Questions may or may not be independent of the preceding ones. You should attempt to solve all the questions.

1a. By using the axioms of set theory show that a function (as it is defined during the lectures) is a set.

1b. Show that the collection of functions from a set \( X \) into a set \( Y \) is a set.

1c. Let \( f: X \rightarrow Y \) be a function. Recall that if \( A \) is a subset of \( Y \), \( f^{-1}(A) \) is defined to be as the set \( \{ x \in X : f(x) \in A \} \). If \( A \) and \( B \) are two subsets of \( Y \), what is the relationship (in terms of inclusion and equality) between the sets \( f^{-1}(A) \cup f^{-1}(B) \), \( f^{-1}(A \cup B) \), \( f^{-1}(A) \cap f^{-1}(B) \) and \( f^{-1}(A \cap B) \)?

1d. Show that the same kind of relations hold for an infinite family \( (A_i)_{i \in I} \) of subsets of \( Y \) (instead of just two subsets).

1e. Let \( f: X \rightarrow Y \) be a function and \( A \) a subset of \( Y \). Is \( A = f(f^{-1}(A)) \) always? If so prove it, otherwise give a counterexample.

1f. Let \( f: X \rightarrow Y \) be a function and \( A \) a subset of \( X \). Is \( A = f^{-1}(f(A)) \) always? If so prove it, otherwise give a counterexample.

2a. Show that if \( f \circ g \) (composition of two functions) is one-to-one, then \( g \) is one-to-one. Give an example where \( f \circ g \) is one-to-one, but \( f \) is not.

2b. State a statement similar to the one above with “onto” instead of “one-to-one”.

2c. Let \( f: X \rightarrow X \) be a function. Show that if \( f^n \) is a bijection for some positive integer \( n \), then \( f \) is also a bijection. (Recall that \( f^n \) is \( f \) composed with itself \( n \) times).

3. If \( X \) and \( Y \) are two topological spaces, a map \( f: X \rightarrow Y \) is called **continuous**, if for any open subset \( U \) of \( Y \), \( f^{-1}(U) \) is an open subset of \( X \).

3a. Let \( X \) be a topological space and \( f: X \rightarrow \mathbb{R} \) a function. Show that if the inverse image \( f^{-1}(a,b) \) of an open interval \( (a,b) \) of \( \mathbb{R} \) is open in \( X \), then \( f \) is continuous. (\( \mathbb{R} \) is endowed with the usual topology generated by the open intervals).

3b. By using 3a, show that the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = x^2 \) is continuous.

3c. Give an example of a noncontinuous function from \( \mathbb{R} \) into \( \mathbb{R} \).

3d. Assume the only open subsets of \( X \) are \( \emptyset \) and \( X \). How can one characterize the continuous functions from \( X \) into a topological space \( Y \)?

3e. Find an infinite topological space \( X \) such that every function \( f \) from \( X \) into any topological space \( Y \) is continuous.
3f. As $\mathbb{Z}$ is a subset of $\mathbb{R}$ and $\mathbb{R}$ is a topological space (in the usual sense), we may endow $\mathbb{Z}$ with the induced topology. What are the open subsets of $\mathbb{Z}$ with this topology?

4a. Let $x$ be any set. Is there always a bijection between the sets $\mathbb{N}$ and $\mathbb{N} \cup \{x\}$?

4b. Find a bijection between $\mathbb{N}$ and $\mathbb{Q}$. If $f$ is this bijection, what is $f(10)$?

4c. Is there a bijection $f$ between $\mathbb{N}$ and $\mathbb{Q}$ that preserves the order, i.e. for which $x < y$ implies $f(x) < f(y)$?

5. Let $X$ be any set. Show that there is no bijection between $X$ and the set $P(X)$ of subsets of $X$. (Hint: Assume there is such a bijection, call it $f$:

\[ f: X \longrightarrow \mathcal{P}(X). \]

Let $A = \{ x \in X : x \notin f(x) \}$. Let $a \in X$ be such that $f(a) = A$. Try to decide whether $a \in A$.

Did you, by any chance, show more than what I have asked?