Set Theory
(Orders)

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An **preorder** on a nonempty set X is a subset A of $X \times X$ such that

a) For any $x \in X$, $(x, x) \notin A$.

b) For any $x, y, z \in X$, if $(x, y) \in A$ and $(y, z) \in A$, then $(x, z) \in A$.

From now on A denotes a preorder on X. One most often writes $x <_A y$, or even $x < y$, instead of $(x, y) \in A$.

1. Let B be a set. For $U, V \subseteq B$, show that the relation $U \subset V$ defines a preorder on the set $\wp(B)$ of subsets of B.

2. Show that if $(x, y) \in A$ for $x, y \in X$, then $(y, x) \notin A$.

3. Show that if (x, y) and $(y, z) \in A$ for $x, y, z \in X$, then $(z, x) \notin A$.

4. A **minimal element** for the preorder is an element x such that $(y, x) \notin A$ for any $y \in X$. Show that if X is a finite set, then X has minimal and maximal elements.

5. Give an example of a preorder with two maximal elements but without minimal elements.

6. How many preorders are there on a set of two elements?

7. How many preorders are there on a set of three elements?

Let $(X, <)$ and $(Y, \prec)$ be two preorders. A **morphism** from $(X, <)$ into $(Y, \prec)$ is a map f from X into Y such that for any $x_1, x_2 \in X$, $x_1 < x_2$ iff $f(x_1) \prec f(x_2)$.

8. Show that a morphism is necessarily one-to-one.

9. Show that the composition of morphisms is a morphism.

A morphism f from $(X, <)$ into $(Y, \prec)$ is called an **isomorphism** if the map is onto. The identity map is clearly an isomorphism. The preorders $(X, <)$ and $(Y, \prec)$ are then said to be **isomorphic**.

9. Show that the inverse of an isomorphism from $(X, <)$ into $(Y, \prec)$ is an isomorphism from $(Y, \prec)$ into $(X, <)$

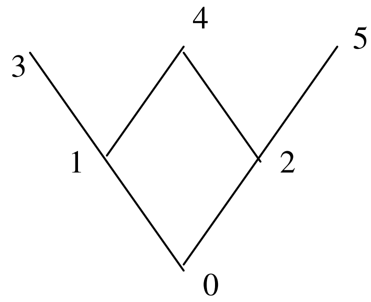
10. How many nonisomorphic preorders are there on a set of three elements?

11. How many nonisomorphic preorders are there on a set of four elements?

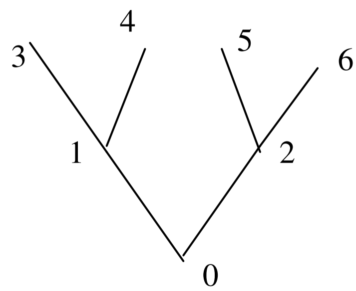
12. Let $\sigma \in \mathrm{Sym}(B)$. Show that σ gives rise (naturally) to an automorphism of the preorder $(\wp(B), \subset)$. Conversely, show that every automorphism of the preorder $(\wp(B), \subset)$ is of this form.

An **automorphism** of a preorder is an isomorphism from the preorder into itself.

12. Find all the automorphisms of the following preorder



13. Find all the automorphisms of the following preorder



A **total order** is an order where any two elements are comparable, i.e. for any x, $y \in X$, either $x < y$ or $x = y$ or $y < x$ (only one of the relations may hold). For example, the natural orders on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are total orders.

14. Show that any two total orders on a finite set X are isomorphic.

15. Find all the automorphisms of $(\mathbb{N}, <)$.

16. Find all the automorphisms of $(\mathbb{Z}, <)$.

17. Show that the total orders $(\mathbb{N}, <)$ and $(\mathbb{N} \setminus \{0\}, <)$ are isomorphic.

18. Show that the total orders $(\mathbb{Z}, <)$ and $(\mathbb{Z} \setminus \{5\}, <)$ are isomorphic.

19. Show that the total orders $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are not isomorphic.

20. Define a total order on the set $\mathbb{N} \cup \{\infty\}$ by extending the total order of $\mathbb{N}$ by adding $n < \infty$ for all $n \in \mathbb{N}$. Show that $\mathbb{N} \cup \{\infty\}$ and $\mathbb{N}$ are not isomorphic.