1. Find $\bigcup M$ and $\bigcap M$ where $M$ is the set whose elements are
   1a. subsets of $\mathbb{R}$ that contain the interval $(0, 1)$.
   1b. subsets of $\mathbb{R}$ that contain the integers.
   1c. intervals of the form $(1/2 - \varepsilon, 1/2 + \varepsilon)$ for $\varepsilon > 0$.
   1d. intervals of the form $(1/2 - \varepsilon, 1/2 + \varepsilon^2)$ for $\varepsilon > 0$.
   1e. intervals of the form $(1/n, x)$ for $n \in \mathbb{N} \setminus \{0,1,2,3,4\}$ and $2/3 < x \leq 1$.
   1f. intervals of the form $(1/2^m, 2^n)$ for $n \in \mathbb{N}$.
   1g. subsets of a given set.
   1h. subsets of $\mathbb{R}$ that contain all rational numbers.

2. For $A$ and $B$ two sets, let $A \Delta B = (A \cup B) \setminus (A \cap B)$.
   2a. Show that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
   2b. Show that $A \Delta A = \emptyset$.
   2c. What can you say about $A \Delta B$ if $A \subseteq B$? Does the reverse statement hold?
   2d. Show that $A \Delta B = B \Delta A$ for all $A$ and $B$.
   2e. Show that $A \Delta \emptyset = A$ for all $A$.
   2f. Show that $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ for all $A$, $B$ and $C$.

3. Let $X$, $Y$ and $Z$ be three sets. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We define a function $g \circ f : X \to Z$ by the rule $(g \circ f)(x) = g(f(x))$ for all $x \in X$. The function $g \circ f$ is called the composition of $f$ and $g$.
   3a. Let $\text{Id}_Y : Y \to Y$ be defined by $\text{Id}_Y(y) = y$ for all $y \in Y$. The function $\text{Id}_Y$ is called the identity function on $Y$. Show that $\text{Id}_Y \circ f = f$.
   3b. Show that $g \circ \text{Id}_Y = g$.
   3c. (Associativity of the composition) Let $X$, $Y$, $Z$ and $T$ be sets. Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to T$ be three functions. Note that $(h \circ g) \circ f$ and $h \circ (g \circ f)$ both make sense and that they are both functions from $X$ into $T$. Show that $(h \circ g) \circ f = h \circ (g \circ f)$.

4a. Find three different functions from $\mathbb{N}$ into $\mathbb{N}$ such that $f^2 = \text{Id}_\mathbb{N}$. (Recall that $f^2$ stands for $f \circ f$).
   4b. Find two different functions from $\mathbb{N}$ into $\mathbb{N}$ such that $f^3 = \text{Id}_\mathbb{N}$.

5. A function $f : X \to Y$ is said to be one-to-one if, for any $x_1, x_2 \in X$, $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.
   5a. Show that if $f : X \to Y$ and $g : Y \to Z$ are one-to-one, then so is $g \circ f$. 


5b. Let \( f : X \to Y \) be a function. Show that \( f \) is one-to-one if and only if there is a function \( g : Y \to X \) such that \( g \circ f \) is one-to-one if and only if there is a function \( g : Y \to X \) such that \( g \circ f = \text{Id}_Y \).

6. A function \( f : X \to Y \) is said to be onto if, for any \( y \in Y \), there is an \( x \in X \) such that \( f(x) = y \).
   
   6a. Show that if \( f : X \to Y \) and \( g : Y \to Z \) are onto, then so is \( g \circ f \).

6b. Let \( f : X \to Y \) be a function. Show that \( f \) is onto if and only if there is a function \( g : Y \to X \) such that \( g \circ f \) is onto and only if there is a function \( g : Y \to X \) such that \( f \circ g \) is onto.

7. A function \( f : X \to Y \) is said to be a bijection if it is both onto and one-to-one.

7a. Show that \( \text{Id}_X \) is a bijection.

7b. Find a one-to-one function which is not a bijection.

7c. Find an onto function which is not a bijection.

7d. Show that if \( f : X \to Y \) and \( g : Y \to Z \) are bijections, then so is \( g \circ f \).

7e. Let \( f : X \to Y \) be a function. Show that \( f \) is a bijection if and only if there is a function \( g : Y \to X \) such that \( f \circ g = \text{Id}_Y \) and \( g \circ f = \text{Id}_X \).

7f. Show that a function \( g \) as above is unique. This function is called the inverse of \( f \) and is denoted by \( f^{-1} \).

7g. Find a bijection from \( \mathbb{N} \) into \( \mathbb{N} \) such that \( f^n \neq \text{Id}_\mathbb{N} \) for all \( n \in \mathbb{N} \setminus \{0\} \).

8. Let \( X \) be a set. the set of bijections from \( X \) into \( X \) is denoted by \( \text{Sym}(X) \).

8a. Show that \( \text{Sym}(X) \) has \( n! \) elements if \( X \) has \( n \) elements.

8b. Let \( X = \{1,2\} \). Find all the elements of \( \text{Sym}(X) \).

8c. Let \( X = \{1,2,3\} \). Find all the elements of \( \text{Sym}(X) \).

8d. Let \( X = \{1,2,3,4\} \). Find all the elements of \( \text{Sym}(X) \).

8e. Show that \( \text{Sym}(X) \) has the following three properties:
   
   (i) For all \( f, g, h \in \text{Sym}(X) \), \( f \circ (g \circ h) = (f \circ g) \circ h \).

   (ii) For all \( f \in \text{Sym}(X) \), \( f \circ \text{Id}_X = \text{Id}_X \circ f = f \).

   (iii) For any \( f \in \text{Sym}(X) \), there exists a \( g \in \text{Sym}(X) \) such that \( f \circ g = g \circ f = \text{Id}_X \).

8f. Show that \( \text{Id}_X \) is the only element of \( \text{Sym}(X) \) that satisfies 8f(ii).

8g. Show that, given \( f \in \text{Sym}(X) \), the element \( g \in \text{Sym}(X) \) as in 8f(iii) is unique. In case you did not show it before, show that this element \( g \) is in fact \( f^{-1} \).