Naive Set Theory

Work for Nesin Foundation
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1. Find $\cup\mathcal{M}$ and $\cap\mathcal{M}$ where $\mathcal{M}$ is the set whose elements are

1a. subsets of $\mathbb{R}$ that contain the interval $(0, 1)$.

1b. subsets of $\mathbb{R}$ that contain the integers.

1c. intervals of the form $(1/2 - \varepsilon, 1/2 + \varepsilon)$ for $\varepsilon > 0$.

1d. intervals of the form $(1/2 - \varepsilon, 1/2 + \varepsilon^2)$ for $\varepsilon > 0$.

1e. intervals of the form $(1/n, x)$ for $n \in \mathbf{N} \setminus \{0,1,2,3,4\}$ and $2/3 < x \le 1$.

1f. intervals of the form $(1/2^n, 2^n)$ for $n \in \mathbf{N}$.

1g. subsets of a given set.

1e. subsets of $\mathbb{R}$ that contain all rational numbers.

2. For A and B two sets, let $A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$.

2a. Show that $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$.

2b. Show that $A \,\Delta\, A = \varnothing$.

2c. What can you say about $A \,\Delta\, B$ if $A \subseteq B$? Does the reverse statement hold?

2d. Show that $A \,\Delta\, B = B \,\Delta\, A$ for all A and B.

2e. Show that $A \,\Delta\, \varnothing = A$ for all A.

2f. Show that $(A \,\Delta\, B) \,\Delta\, C = A \,\Delta\, (B \,\Delta\, C)$ for all A, B and C.

3. Let X, Y and Z be three sets. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We define a function $g \circ f : X \to Z$ by the rule $(g \circ f)(x) = g(f(x))$ for all $x \in X$. The function $g \circ f$ is called the **composition** of f and g.

3a. Let $\mathrm{Id}_Y : Y \to Y$ be defined by $\mathrm{Id}_Y(y) = y$ for all $y \in Y$. The function Id_Y is called the **identity function** on Y. Show that $\mathrm{Id}_Y \circ f = f$.

3b. Show that $g \circ \mathrm{Id}_Y = g$.

3c. (**Associativity of the composition**) Let X, Y, Z and T be sets. Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to T$ be three functions. Note that $(h \circ g) \circ f$ and $h \circ (g \circ f)$ both make sense and that they are both functions from X into T. Show that $(h \circ g) \circ f = h \circ (g \circ f)$.

4a. Find three different functions from $\mathbb{N}$ into $\mathbb{N}$ such that $f^2 = \mathrm{Id}_{\mathbf{N}}$. (Recall that f^2 stands for $f \circ f$).

4b. Find two different functions from $\mathbb{N}$ into $\mathbb{N}$ such that $f^3 = \mathrm{Id}_{\mathbf{N}}$.

5. A function $f : X \to Y$ is said to be **one-to-one** if, for any $x_1, x_2 \in X$, $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

5a. Show that if $f : X \to Y$ and $g : Y \to Z$ are one-to-one, then so is $g \circ f$.

5b. Let $f : X \rightarrow Y$ be a function. Show that f is one-to-one if and only there is a function $g : Y \rightarrow X$ such that $g \circ f$ is one-to-one if and only there is a function $g : Y \rightarrow X$ such that $g \circ f = \mathrm{Id}_X$.

6. A function $f : X \rightarrow Y$ is said to be **onto** if, for any $y \in Y$, there is an $x \in X$ such that $f(x) = y$.

6a. Show that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are onto, then so is $g \circ f$.

6b. Let $f : X \rightarrow Y$ be a function. Show that f is onto if and only there is a function $g : Y \rightarrow X$ such that $f \circ g$ is onto if and only there is a function $g : Y \rightarrow X$ such that $f \circ g = \mathrm{Id}_Y$.

7. A function $f : X \rightarrow Y$ is said to be a **bijection** if it is both onto and one-to-one.

7a. Show that Id_X is a bijection.

7b. Find a one-to-one function which is not a bijection.

7c. Find an onto function which is not a bijection.

7d. Show that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are bijections, then so is $g \circ f$.

7e. Let $f : X \rightarrow Y$ be a function. Show that f is a bijection if and only there is a function $g : Y \rightarrow X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

7f. Show that a function g as above is unique. This function is called the **inverse** of f and is denoted by f^{-1}.

7g. Find a bijection from **N** into **N** such that $f^n \neq \mathrm{Id}_{\mathbf{N}}$ for all $n \in \mathbf{N} \setminus \{0\}$.

8. let X be a set. the set of bijections from X into X is denoted by $\mathrm{Sym}(X)$.

8a. Show that $\mathrm{Sym}(X)$ has $n!$ elements if X has n elements.

8b. Let $X = \{1,2\}$. Find all the elements of $\mathrm{Sym}(X)$.

8c. Let $X = \{1,2,3\}$. Find all the elements of $\mathrm{Sym}(X)$.

8d. Let $X = \{1,2,3,4\}$. Find all the elements of $\mathrm{Sym}(X)$.

8e. Show that $\mathrm{Sym}(X)$ has the following three properties:

(i) For all $f, g, h \in \mathrm{Sym}(X)$, $f \circ (g \circ h) = (f \circ g) \circ h$.

(ii) For all $f \in \mathrm{Sym}(X)$, $f \circ \mathrm{Id}_X = \mathrm{Id}_X \circ f = f$.

(iii) For any $f \in \mathrm{Sym}(X)$, there exists a $g \in \mathrm{Sym}(X)$ such that $f \circ g = g \circ f = \mathrm{Id}_X$.

8f. Show that Id_X is the only element of $\mathrm{Sym}(X)$ that satisfies 8f(ii).

8g. Show that, given $f \in \mathrm{Sym}(X)$, the element $g \in \mathrm{Sym}(X)$ as in 8f(iii) is unique. In case you did not show it before, show that this element g is in fact f^{-1}.