Math 111 Final Exam
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Notes. Yes or no answers will not be accepted. Proofs and disproofs are necessary.
Please make full sentences, with a subject and a verb, at least.
Do not use symbols such as $\exists$, $\forall$, $\Rightarrow$, etc.
Write neetly, clearly, understandably etc.

Let F be the set of functions from some open interval of $\mathbb{R}$ containing 0 into $\mathbb{R}$. Note that two functions of f may have different domains. For $f, g \in F$, define the relation $f \equiv g$ by the condition "there is an open interval I containg 0 such that $f(x) = g(x)$ for all $x \in I$, i.e. if $f = g$ on some open interval containing 0".

1. *Show that* $\equiv$ *is an equivalence relation on* F. (*The equivalence class* $[f]$ *of* f *is called the* ***germ*** *of* f *at* 0). (6 pts.)

Proof: Reflexivity: If $f : I \rightarrow \mathbb{R}$ is a function from an open interval I containing 0 into $\mathbb{R}$, then of course $f(x) = f(x)$ for all $x \in I$. Therefore $f \equiv f$.

Symmetry: Let $f : I \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$ be two functions. (Here I and J are two open intervals both containing 0). Assume $f \equiv g$. Then $f(x) = g(x)$ for x in some open interval K containing 0. (Note that K must be a subinterval of I and J because otherwise $f(x)$ and $g(x)$ are not defined). Thus $g(x) = f(x)$ for $x \in K$. Therefore $g \equiv f$.

Transitivity: Let $f : I \rightarrow \mathbb{R}$, $g : J \rightarrow \mathbb{R}$ and $h : K \rightarrow \mathbb{R}$ be three functions in F. (Here I, J and K are three open intervals all containing 0). Suppose $f \equiv g$ and $g \equiv h$. Thus $f(x) = g(x)$ for all x in some open interval A containing 0 and $g(x) = h(x)$ for all x in some open interval B containing 0. (Here $A \subseteq I \cap J$ and $B \subseteq J \cap K$). Then $f(x) = h(x)$ for all $x \in A \cap B$. Since $A \cap B$ is an open interval containing 0, this proves that $f \equiv g$.

2. *Show that one is allowed to define addition and multiplication of elements of* $F/\equiv$ *in a natural way, namely by the rules*

$$[f] + [g] = [f + g] \text{ and } [f][g] = [fg].$$

(10 pts.)

Proof: Note first that if $f : I \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$, then as the domain of the functions $f + g$ and fg we have to take $I \cap J$. Then $(f + g)(x)$ and $(fg)(x)$ can be defined as $f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$.

Suppose $[f] = [f_1]$ and $[g] = [g_1]$, i.e. suppose $f \equiv f_1$ and $g \equiv g_1$. Thus there are open intervals A and B both containing 0 such that $f(x) = f_1(x)$ for all $x \in A$ and $g(x) = g_1(x)$ for all $x \in B$. Thus

$$(f + f_1)(x) = f(x) + f_1(x) = g(x) + g_1(x) = (g + g_1)(x)$$

and

$$(ff_1)(x) = f(x)f_1(x) = g(x)g_1(x) = (gg_1)(x)$$

for all $x \in A \cap B$. Since $A \cap B$ is an open interval containing 0, this proves that $f + f_1 \equiv g + g_1$ and $ff_1 \equiv gg_1$. Thus $[f + f_1] = [g + g_1]$ and $[ff_1] \equiv [gg_1]$.

3. *Show that, with the above operations* $F/\equiv$ *is a ring with identity, i.e.*, (Prove only A2, M1 and M2.)

A1. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all $\alpha, \beta, \gamma \in F/\equiv$.
A2. *There is an element* $0 \in F/\equiv$ *such that* $0 + \alpha = \alpha + 0 = \alpha$ *for all* $\alpha \in F/\equiv$.
A3. *For all* $\alpha \in F/\equiv$ *there is a* $\beta \in F/\equiv$ *such that* $\alpha + \beta = \beta + \alpha = 0$. (*Here* 0 *is as in* A2).
A4. $\alpha + \beta = \beta + \alpha$ *for all* $\alpha, \beta \in F/\equiv$.
M1. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *for all* $\alpha, \beta, \gamma \in F/\equiv$.
M2. *There is an element* $1 \in F/\equiv$ *such that* $1\cdot\alpha = \alpha\cdot 1 = \alpha$ *for all* $\alpha \in F/\equiv$.
M3. $\alpha\beta = \beta\alpha$ *for all* $\alpha, \beta \in F/\equiv$.
D. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ *for all* $\alpha, \beta, \gamma \in F/\equiv$.
(12 pts.)

Proof of A2: Let c_0 denote the constant zero-function from $\mathbb{R}$ into $\mathbb{R}$. Thus $c_0(x) = 0$ for all $x \in \mathbb{R}$. Since $\mathbb{R}$ is an interval containing 0, the constant zero-function c_0 is an element of F. Now we can consider the class $[c_0] \in F/\equiv$ of the zero-function $c_0 \in F$.

Let $\alpha \in F/\equiv$. Thus $\alpha = [f]$ for some function $f : I \to \mathbb{R}$. (Here I is an open interval containing 0). Let us compute $[c_0] + [f]$.

It should be clear that $c_0 + f \equiv f$ because for any $x \in I$, $(c_0 + f)(x) = c_0(x) + f(x) = c_0 + f(x) = f(x)$. Thus $[c_0] + \alpha = [c_0] + [f] = [c_0 + f] = [f]$.

Similarly $\alpha + [c_0] = \alpha$.

Therefore $[c_0]$ is the neutral element of $F/\equiv$ for the addition. We may take the element 0 of $F/\equiv$ that we were looking to be equal to $[c_0]$. (Note that this $0 \in F/\equiv$ that we have just defined is not the number $0 \in \mathbb{R}$).

Proof of M1. Let $\alpha, \beta, \gamma \in F/\equiv$. Let $f, g, h \in F$ be such that $\alpha = [f]$, $\beta = [g]$, $\gamma = [h]$. Then $(\alpha\beta)\gamma = ([f][g])[h] = [fg][h] = [(fg)h] = [f(gh)] = [f][gh] = [f]([g][h]) = \alpha(\beta\gamma)$.

Proof of M2: Let c_1 denote the constant one-function from $\mathbb{R}$ into $\mathbb{R}$. Thus $c_1(x) = 1$ for all $x \in \mathbb{R}$. Since $\mathbb{R}$ is an interval containing 1, the constant one-function c_1 is an element of F. Now we can consider the class $[c_1] \in F/\equiv$ of the one-function $c_1 \in F$.

Let $\alpha \in F/\equiv$. Thus $\alpha = [f]$ for some function $f : I \to \mathbb{R}$. (Here I is an open interval containing 1). Let us compute $[c_1][f]$.

It should be clear that $c_1 f \equiv f$ because for any $x \in I$, $(c_1 f)(x) = c_1(x)f(x) = 1.f(x) = f(x)$. Thus $[c_1]\alpha = [c_1][f] = [c_1 f] = [f]$.

Similarly $\alpha[c_1] = \alpha$.

Therefore $[c_1]$ is the neutral element of $F/\equiv$ for the multiplication. We may take the element 1 of $F/\equiv$ that we were looking to be equal to $[c_1]$. (Note that this $1 \in F/\equiv$ that we have just defined is not the number $1 \in \mathbb{R}$).

4. *Find the set of invertible elements of the ring* $F/\equiv$, *i.e. find*

$$(F/\equiv)^* = \{\alpha \in F/\equiv : \alpha\beta = 1 \text{ for some } \beta \in F/\equiv\}.$$

(6 pts.)

Solution: Let $\alpha \in (F/\equiv)^*$. Let $\beta \in F/\equiv$ be such that $\alpha\beta = 1$. Recall that, here, $1 = [c_1]$. Let f and g be elements of F for which $\alpha = [f]$ and $\beta = [g]$. Then $[c_1] = 1 = \alpha\beta = [f][g] = [fg]$ and so $fg \equiv c_1$. Therefore there is an open interval I containing 0 such that $(fg)(x) = c_1(x)$ for all $x \in I$. Computing, we find that $f(x)g(x) = 1$ for all $x \in I$. In particular f never assumes the 0 value in an open interval of 0.

Conversely, we will show that if $f \in F$ never assumes the 0 value in an open interval of 0, then $[f] \in (F/\equiv)^*$. Indeed, assume that $f(x) \neq 0$ for all $x \in I$, where I is an open interval containing 0. Define $g : I \to \mathbb{R}$ by the rule $g(x) = 1/f(x)$. Then clearly $(fg)(x) = f(x)g(x) = 1 = c_1(x)$ for all $x \in I$. It follows that $fg \equiv c_1$. Therefore $[f][g] = [fg] = [c_1] = 1$ and so $[f] \in (F/\equiv)^*$.

We proved that $(F/\equiv)^* = \{[f] \in F/\equiv : f$ never assumes the value 0 in an open interval containing 0$\}$.

5. *Does the ring* $F/\equiv$ *have nonzero zerodivisors, i.e. are there nonzero* $\alpha, \beta \in F/\equiv$ *such that* $\alpha\beta = 0$? (6 pts.)

Solution: Yes. Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = x$ and $g : \mathbb{R} \to \mathbb{R}$ by the rule $g(x) = 0$ if $x \neq 0$ and $g(x) = 1$ if $x = 1$. Then fg is clearly the zero-function c_0. Hence $[f][g] = [fg] = [c_0] = 0$. But $[f] \neq 0$ and $[g] \neq 0$ because neither f nor g is equal to the zero-function in an open interval of 0.

6. Does the ring $F/\equiv$ have nonzero nilpotent elements, i.e. is there a nonzero $\alpha \in F/\equiv$ such that $\alpha^n = 0$ for some positive natural number n? (6 pts.)

Solution: No. Assume $\alpha \in F/\equiv$ is such that $\alpha^n = 0$ for some positive natural number n. We will show that $\alpha = 0$. Recall that, here, 0 means $[c_0]$. Let $f \in F$ be such that $\alpha = [f]$. Then $[c_0] = 0 = \alpha^n = [f]^n = [f^n]$. (Note that, here, f^n means $f \ldots f$, the product of f with itself n times, i.e. $(f^n)(x) = f(x)^n$ for all x in the domain of f). Thus $f^n \equiv c_0$ and so there is an open interval I containing 0 such that $f^n(x) = c_0(x)$ for all $x \in I$. Thus $f(x)^n = f^n(x) = c_0(x) = 0$ for all $x \in I$. Since $f(x) \in \mathbb{R}$, this implies that $f(x) = 0$ for all $x \in I$. Therefore $f \equiv c_0$ and so $\alpha = [f] = 0$.

7. *Show that there is an embedding* (*one-to-one function*) *of* $(\mathbb{R}, +, \times)$ *into F that respects the addition and multiplication.* (12 pts.)

Proof: For $a \in \mathbb{R}$, let c_a denote the constant function that takes only the value a, i.e. the function $c_a : \mathbb{R} \to \mathbb{R}$ is defined by $c_a(x) = a$ for all $x \in \mathbb{R}$. Define the function $i : \mathbb{R} \to F/\equiv$ by the rule $i(a) = [c_a]$ for all $a \in \mathbb{R}$. We need to show that i is one to one and that $i(a + b) = i(a) + i(b)$ and $i(ab) = i(a)i(b)$ for all $a, b \in \mathbb{R}$.

Proof that *i* is one-to-one. Assume $i(a) = i(b)$. Thus $[c_a] = [c_b]$. This means that there is an interval I containing 0 such that $c_a(x) = c_b(x)$ for all $x \in I$. Since $0 \in I$, we can apply this last equality to $x = 0$ (or to any element x in I), to get $a = c_a(x) = c_b(x) = b$.

Proof that *i* is additive. We first check that $c_a + c_b = c_{a+b}$. Indeed, for any $x \in \mathbb{R}$,

$$(c_a + c_b)(x) = c_a(x) + c_b(x) = a + b = c_{a+b}(x).$$

This proves the equality $c_a + c_b = c_{a+b}$. Hence,

$$i(a + b) = [c_{a+b}] = [c_a + c_b] = [c_a] + [c_b] = i(a) + i(b).$$

Proof that *i* is multiplicative. We first check that $c_a c_b = c_{ab}$. Indeed, for any $x \in \mathbb{R}$,

$$(c_a c_b)(x) = c_a(x)c_b(x) = ab = c_{ab}(x).$$

This proves the equality $c_a c_b = c_{ab}$. Hence,

$$i(ab) = [c_{ab}] = [c_a c_b] = [c_a][c_b] = i(a)i(b).$$

8. *For* $\alpha \in F/\equiv$, *define* $v_0(\alpha) = f(0)$ *for some* $f \in \alpha$. *Show that* v_0 *is a well-defined function from* $F/\equiv$ *onto* $\mathbb{R}$. (12 pts.)

Proof: Suppose $f, g \in \alpha$. Then $f \equiv g$, i.e. there is an open interval I containing 0 such that $f(x) = g(x)$ for all $x \in I$. Since $0 \in I$, in particular, $f(0) = g(0)$. This shows that v_0 is a well-defined function.

Let us now prove that v_0 is onto. Indeed if $y \in \mathbb{R}$, then $v_0(c_y) = c_y(0) = y$.

9. *Show that the relation* $\approx$ *on* $F/\equiv$ *defined by* $\alpha \approx \beta$ *if and only if* $v_0(\alpha) = v_0(\beta)$ *is an equivalence relation.*(14 pts.)

Proof: This is a complete triviality: All follows from the definition

$$\alpha \approx \beta \text{ if and only if } v_0(\alpha) = v_0(\beta).$$

10. *Can you put a natural algebraic structure on* $(F/\equiv)/\approx$? *Can you find its structure*? (*I.e. does it look like a known algebraic structure*?) (16 pts.)

Solution: Yes! $(F/\equiv)/\approx$ looks like $\mathbb{R}$. Define the function $j : (F/\equiv)/\approx \rightarrow \mathbb{R}$ by the rule,

$$j([\alpha]) = v_0(\alpha).$$

This function is well-defined because if $[\alpha] = [\beta]$ then $\alpha \approx \beta$ and so $v_0(\alpha) = v_0(\beta)$ by definition of the relation $\approx$.

The function is one-to-one because if $j([\alpha]) = j([\beta])$, then $v_0(\alpha) = v_0(\beta)$ and so $\alpha \approx \beta$ and $[\alpha] = [\beta]$.

The function is onto because for any $y \in \mathbb{R}$, $j(i(y)) = v_0([c_y]) = c_y(0) = y$.

Thus there is a "natural" bijection between the set $(F/\equiv)/\approx$ and $\mathbb{R}$. We can define an "addition" and a "multiplication" to make $(F/\equiv)/\approx$ look more like $\mathbb{R}$. Indeed, define,

$$[\alpha] + [\beta] = [\alpha + \beta]$$

and

$$[\alpha][\beta] = [\alpha\beta].$$

It is easy to check that then j respects the addition and multiplication.