Math 111

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Homework
on Functions (According to Naïve Set Theory)
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Let X and Y be two sets. Naively speaking[1], a function from X into Y is a "rule" that associates to **each** element x of X a **unique** element y of Y. If we denote by f a function from X into Y, the element y of Y that is associated to the element x of X is denoted by $f(x)$. The element $f(x)$ of Y is called the *value* of the function f at x.

Example 1. Let $X = Y = \mathbb{N}$, the set of natural numbers, and let us associate to a natural number x, the natural number $2x + 1$. If we denote by f this function, we have $f(x) = 2x + 1$. For example, $f(3) = 7$, $f(f(3)) = f(7) = 15$.

Example 2. The rule that associates $x/2$ to a natural number x is **not** a function from $\mathbb{N}$ into $\mathbb{N}$, because $x/2$ is not always a natural number. On the other hand the same rule gives rise to a function from $\mathbb{N}$ into the set $\mathbb{Q}$ of rational numbers.

Example 3. The rule that associates to each $x \in \mathbb{N}$, a real number y such that $y^2 = x$ is **not** a function from $\mathbb{N}$ into $\mathbb{R}$, because, except for $x = 0$, there are two distinct solutions of the equation $y^2 = x$. In other words, the value y is not always unique.

1. How many functions are there from a set of n elements into a set of m elments?

If f is a function from X into Y and if A is a subset of X, we denote by $f(A)$ the set values of f at the elements of A. More formally,

$$f(A) = \{y \in Y : y = f(a) \text{ for some } a \text{ in } A\}.$$

2. Let f be as in Example 1,

2a. Find $f(\mathbb{N})$.

2b. Find $f(f(\mathbb{N}))$.

2c. Find $f(f(f(\mathbb{N})))$.

2d. Find $f(f\ldots(f(\mathbb{N}))\ldots)$. (Here there are n f's. We denote this set by $f^n(\mathbb{N})$).

Let f be a function from X into Y and let A and B be two subsets of X.

3. Show that if $A \subseteq B$, then $f(A) \subseteq f(B)$.

4. Show that $f(A \cup B) = f(A) \cup f(B)$. The same equality holds of course for any finite set of subsets.

5. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$. The same inclusion holds of course for any finite set of subsets.

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[1] Later on, we will introduce functions formally. As everything else in formal set theory, a function will be a set.

6. Show that $f(A \cap B)$ may be different from $f(A) \cap f(B)$. (You have to find examples of X, Y, f, A and B).

7. What can you say about the relationship between $f(A \setminus B)$ and $f(A) \setminus f(B)$?

8. Show that $f(\varnothing) = \varnothing$.

Let $(A_i)_{i\in I}$ be a family of subsets of X. Naively speaking, this means that I is a set and that for each element i of I, a subset A_i of X is given.

9. Show that $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$. This is generalization of question #4.

10. Show that $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$. This is generalization of question #5.

From now on we assume that f is a function from the set X into itself, i.e. f is a function from X into X. We say that a subset A of X is *f-closed* if $f(A) \subseteq A$. The sets X and $\varnothing$ are of course f-closed.

11. Show that if A is an f-closed subset of X, then so is $f(A)$.

12. Show that $\bigcup_{n\in \mathbf{N}} f^n(A)$ is an f-closed subset of X. (By convention, $f^0(A) = A$. For $n > 0$, the meaning of $f^n(A)$ should be clear from question #2d.

13. Is $\bigcap_{n\in \mathbf{N}} f^n(A)$ an f-closed subset for any function f and subset A of X?

14. Show that the intersection of f-closed sets is f-closed. In other words, show that if each A_i is an f-closed subset of X for each $i \in I$, then $\bigcap_{i\in I} f(A_i)$ is also an f-closed subset of X.

15. Let A be a subset of X. Show that the intersection of all the f-closed subsets of X that contain A is the unique smallest f-closed subset of X that contains A. You have to show that:

15a. The intersection of all the f-closed subsets of X that contain A is an f-closed subset of X.

15b. The intersection of all the f-closed subsets of X that contain A contains A.

15c. If B is an f-closed subset of X that contains A, then B contains the intersection of all the f-closed subsets of X that contain A.

Let A^* denote this unique subset.

16. Show that $A^* = \bigcup_{n\in \mathbf{N}} f^n(A)$.

17. Let f be as in Example 1. Show that $\{0\}^* = \{2^n - 1 : n \in \mathbb{N}\}$.

18. Let f be as in Example 1. Find an equality as above for $\{1\}^*$.

19. Let f be as in Example 1. Find an equality as above for $\{2\}^*$.

20. Show that the intersection of f-closed sets is f-closed.

21. Let A be a subset of X. Show that the union of all the f-closed subsets of A is the unique largest f-closed subset of A. Let A° denote this unique set.

22. Let A be the set of natural numbers not divisible by 3. What is A°?