Introduction to Abstract Mathematics
Final Exam
1996-7

Ali Nesin
Bilgi University

Notation: The letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote respectively the set of natural numbers, integers, rational numbers and real numbers. Recall that the last three are groups under addition. If X is a set, $\wp(X)$ denotes the set of subsets of a set.

Note: You should attempt to solve all the questions. An answer with no proof or justification will not be accepted.

Some questions may depend on the previous ones.

1a. (5 pts.) Is the set of functions from $\mathbb{R}$ into $\mathbb{R} \setminus \{0\}$ a group under the multiplication of functions (If f and g are functions from $\mathbb{R}$ into $\mathbb{R}$, the function fg is defined to be the function whose value at x is $f(x)g(x)$; i.e. $(fg)(x) = f(x)g(x)$)?

1b. (5 pts.) Is the set of functions from $\mathbb{R}$ into the open interval (1/2,2) a group under the multiplication of functions?

1c. (5 pts.) Is the set of functions from $\mathbb{R}$ into $\mathbb{N}$ a group under the multiplication of functions? What about under the addition of functions?

2a. (10 pts.) For two sets A and B, define $A \Delta B$ to be $(A \cup B) \setminus (A \cap B)$. Show that the set of subsets $\wp(X)$ of a set X is a group of exponent 2 under this operation Δ (which is called "the symmetric difference").

2b. (10 pts.) Let $f : X \to Y$ be any function. Show that the map $g : \wp(Y) \to \wp(X)$ defined by $g(A) = f^{-1}(A)$ is a group homomorphism. (Here, $\wp(Y)$ and $\wp(X)$ are considered as groups under Δ).

2c. (5 pts.) Show that if there is a bijection between two sets X and Y, then the above groups $\wp(X)$ and $\wp(Y)$ are isomorphic to each other.

3a. (10 pts.) Show that a subgroup of the group $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer n.

3b. (5 pts.) Given two integers n and m, find x such that $n\mathbb{Z} \cap m\mathbb{Z} = x\mathbb{Z}$ as a function of n and m.

4a. (5 pts.) Let G be the set of sequences of real numbers that converge to some real number. Show that G is a group under addition.

4b. (5 pts.) Let H be the set of sequences of real numbers that converge to 0. Show that H is a subgroup of G.

4c. (5 pts.) Show that the map f: $G \longrightarrow \mathbb{R}$ be defined by $f((a_n)_n) = \lim_{n\to\infty} a_n$ is a group homomorphism.

4d. (5 pts.) What is the kernel and the image of f?

5. (10 pts.) Prove that $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

6. (16 pts.) Find

$$\lim_{n\to\infty} \sin(n)/n,$$
$$\lim_{n\to\infty} n \sin(1/n),$$
$$\lim_{x\to 0} \sin(2x)/3x$$
$$\lim_{n\to\infty} 3^{n+1}/2^{2n}.$$