Complex Analysis

Final Exam
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Open book. (You don’t have to!)
Justify your answers. (You have to!)

1. Find the Taylor series of
   \[ f(z) = z^3 + 2z - 1 \]
   \[ f(z) = 1/z \]
around \( z_0 = 1 \).

2. Find the first four terms of the Laurent series of \( f(z) = 1/\sin z \) around 0.

3. Find \( \int_{|z|=1} \frac{e^z - e^{-z}}{z^4} dz \).

4. Suppose \( f \) is analytic in a domain \( \Omega \), \( \gamma \) is a circle in \( \Omega \) and \( f \) has no zeroes on \( \gamma \). In terms of the zeroes of \( f \) inside \( \gamma \) what is the value of \( \int_{\gamma} \frac{f'}{f} dz \) ?

5. Suppose that \( \Omega \) is a region containing a disc \( D \), \( f \) is a nonconstant analytic function in \( \Omega \) such that \( |f| \) is constant on \( \partial D \). Show that \( f \) has at least one zero inside \( D \).

6. Suppose \( f \) and \( g \) are analytic in a region \( D \). Suppose also that \( f(z)^2 = g(z)^2 \) for \( z \) \in \( D \). What can you say about \( f \) and \( g \)?

7. Suppose \( f \) and \( g \) are entire functions such that \( |f(z)| \leq |g(z)| \) for all \( z \). What can you say about the relationship of \( f \) and \( g \)?

8. Suppose \( f \) is analytic in a domain \( \Omega \) containing the unit disc and \( |f(z)| > 2 \) for all \( |z| = 1 \) and also \( f(0) = 1 \). Show that \( f \) must be equal to zero for some point in the unit disc.

9. Suppose \( f \) is any analytic function in the open unit disc. Show that there must be a sequence \( (z_n)_n \) with \( |z_n| \to 1 \) such that \( f(z_n) \) is bounded.

10. Suppose \( P(z, w) \) is a polynomial in two complex variables. Suppose \( w_0 \) is such that \( P(z, w_0) \) has only simple zeroes (as a polynomial in one variable). Show that the same property holds for all \( w \) near \( w_0 \). Hint: Recall the following weaker version of Rouché’s Theorem: Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( \overline{B(a; R)} \) with no zeroes or poles on the circle \( C(a; R) \). If \( Z_f, Z_g, P_f \) and \( P_g \) denote the number of zeroes and poles of \( f \) and \( g \) inside \( \gamma \) counted according to their multiplicities and if \( |f(z) + g(z)| < |f(z)| \) on \( \gamma \), then \( Z_f - P_f = Z_g - P_g \). Also use the fact that the polynomial function \( f(z, w) \) is uniformly continuous on any compact subset of \( \mathbb{C}^2 \).