Complex Analysis

Final Exam
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Open book. (You don't have to!)
Justify your answers. (You have to!)

1. Find the Taylor series of

$$f(z) = z^3 + 2z - 1$$
$$f(z) = 1/z$$

around $z_o = 1$.

2. Find the first four terms of the Laurent series of $f(z) = 1/\sin z$ around 0.
3. Find $\int_{|z|=1} \frac{e^z - e^{-z}}{z^4} dz$.
4. Suppose f is analytic in a domain Ω, γ is a circle in Ω and f has no zeroes on γ. In terms of the zeroes of f inside γ what is the value of $\int_\gamma \frac{f'}{f} z^p dz$?
5. Suppose that Ω is a region containing a disc D, f is a nonconstant analytic function in Ω such that $|f|$ is constant on ∂D. Show that f has at least one zero inside D.
6. Suppose f and g are analytic in a region D. Suppose also that $f(z)^2 = g(z)^2$ for $z \in D$. What can you say about f and g?
7. Suppose f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all z. What can you say about the relationship of f and g?
8. Suppose f is analytic in a domain Ω containing the unit disc and $|f(z)| > 2$ for all $|z| = 1$ and also $f(0) = 1$. Show that f must be equal to zero for some point in the unit disc.
9. Suppose f is any analytic function in the open unit disc. Show that there must be a sequence $(z_n)_n$ with $|z_n| \to 1$ such that $f(z_n)$ is bounded.
10. Suppose $P(z, w)$ is a polynomial in two complex variables. Suppose w_o is such that $P(z, w_o)$ has only simple zeroes (as a polynomial in one variable). Show that the same property holds for all w near w_o. **Hint:** Recall the following weaker version of Rouché's Theorem: *Suppose f and g are meromorphic in a neighborhood of* $\overline{B}(a;R)$ *with no zeroes or poles on the circle* $C(a; R)$. *If* Z_f, Z_g, P_f *and* P_g *denote the number of zeroes and poles of f and g inside* γ *counted according to their multiplicities and if* $|f(z) + g(z)| < |f(z)|$ *on* γ, *then* $Z_f - P_f = Z_g - P_g$. Also use the fact that the polynomial function $P(z, w)$ is uniformly continuous on any compact subset of $\mathbf{C}^2$.