Math 131 Final
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1. *Let p|q mean $p \wedge \neg q$. Show that the formula $\neg p$ is not tautologically equivalent to a proposition whose only connective is* |. (10 pts.)

Proof: We first show that no proposition with | as the only connective can assume the truth value always 1. Assume not. Let α be such a proposition of smallest length. Write $\alpha = \beta|\gamma$. Then β must always assume the truth value 1, contradicting the fact that α was the smallest such proposition.

We can now show that no proposition with | as the only connective can be tautologically equivalent to $\neg p$.

A proposition $\alpha = \alpha(p, ...)$ with | as the only connective is of the form

$$\beta(p, ...)|\gamma(p, ...)$$

for some shorter propositions β and γ. Here "..." denotes the fact that we may have other atomic propositions in the expressions. Choose α to be tautologically equivalent to $\neg p$ and of minimal length with this property. Since α is tautologically equivalent to $\neg p$ we must have,

a) $\beta(0, ...) = 1$ and $\gamma(0, ...) = 0$ (so that $\alpha(0, ...) = \neg 0 = 1$) and

b) Either $\beta(1, ...) = 0$ or $\gamma(1, ...) = 1$ (so that $\alpha(1, ...) = \neg 1 = 0$).

Let us consider the two subcases of case b separately.

If $\beta(1, ...) = 0$, then, because of condition a, β is itself equivalent to $\neg p$, contradicting the fact that α is of minimal length with this property. Thus $\beta(1, ...) = 1$. Thus β always assumes the truth value 1, contradicting our first fact.

2. *How many words can you write using all the letters of* ABRAKADABRA? (A *must be used* 5 *times*, B *twice etc.*) (10 pts.)

Answer: Let us first replace the five A's by A_1, A_2, A_3, A_4, A_5 in the order of their apparence and the two B's and R's by B_1 and B_2 and R_1 and R_2 in that order. Now we have 11 different letters. We can order them in 11! different ways. Identifying the A's, B's and R's reduces this number to $11!/(5!2!2!) = 11\times10\times9\times8\times7\times6/4 = 11\times10\times9\times2\times7\times6 = 110 \times 18 \times 42 = 1980 \times 42 = 83160$.

3. *Consider the polynomial $(X_1 + X_2 + ... + X_n)^k$ in n variables $X_1, ..., X_n$. When multiplied out, this polynomial is equal to a polynomial of the form*

$$\sum\nolimits_{i_1+i_2+...+i_n=k} a(i_1,...,i_n)X_1^{i_1}X_2^{i_2}...X_n^{i_n}$$

for some $a(i_1, ..., i_n) \in \mathbb{N}$. Here, k runs over all natural numbers and $i_1, i_2, ..., i_n$ run over all natural numbers whose sum is k. Find $a(i_1, ..., i_n)$. Applying the above formula to various values of $X_1, X_2, ..., X_n$ deduce some combinatorial formulas. (20 pts.)

Answer: Write the product $(X_1 + X_2 + ... + X_n)^k$ in the form

$$(X_1 + X_2 + ... + X_n)\,(X_1 + X_2 + ... + X_n)\,...\,(X_1 + X_2 + ... + X_n).$$

Here, there are k factors. To execute the multiplication, from each factor we choose one of the X_i's and multiply these choice to get some monomial of the form $X_1^{i_1}X_2^{i_2}...X_n^{i_n}$. Given i_1, ..., i_n whose sum is k, we have to find out in how many ways we can choose the X_i's from each factor so as to obtain $X_1^{i_1}X_2^{i_2}...X_n^{i_n}$. We have k factors to choose i_1 many X_1's. Thus $X_1^{i_1}$ can be chosen in $\binom{k}{i_1}$ many ways. Now for X_2, there are only $k - i_1$ factors left to choose from.

From these $k - i_1$ factors we have to choose i_2 many X_2's. Hence the number of choice for $X_2{}^{i_2}$ is $\binom{k-i_1}{i_2}$. In a similar way, we find that the number of choices for X_3 is $\binom{k-i_1-i_2}{i_3}$. Hence the monomial $X_1^{i_1}X_2^{i_2}\ldots X_n^{i_n}$ can be chosen in

$$\binom{k}{i_1}\binom{k-i_1}{i_2}\binom{k-i_1-i_2}{i_3}\ldots\binom{k-i_1-i_2-\ldots-i_{n-1}}{i_n}$$

many ways. This can also be written as,

$$\begin{aligned}&\binom{k}{i_1}\binom{k-i_1}{i_2}\binom{k-i_1-i_2}{i_3}\ldots\binom{k-i_1-i_2-\ldots-i_{n-1}}{i_n}\\ &=\frac{k!}{i_1!(k-i_1)!}\frac{(k-i_1)!}{i_2!(k-i_1-i_2)!}\frac{(k-i_1-i_2)!}{i_3!(k-i_1-i_2-i_3)!}\cdots\frac{(k-i_1-i_2-\ldots-i_{n-1})!}{i_n!(k-i_1-i_2-i_3-\ldots-i_n)!}\\ &=\frac{k!}{i_1!i_2!\ldots i_n!}.\end{aligned}$$

Thus

$$a(i_1,\ldots,i_n)=\frac{(i_1+\ldots+i_n)!}{i_1!i_2!\ldots i_n!}.$$

Application. Thus,

$$\begin{aligned}(X_1+\ldots+X_n)^k&=\sum\nolimits_{i_1+i_2+\ldots+i_n=k}a(i_1,\ldots,i_n)X_1^{i_1}X_2^{i_2}\ldots X_n^{i_n}\\ &=\sum\nolimits_{i_1+i_2+\ldots+i_n=k}\frac{(i_1+\ldots+i_n)!}{i_1!i_2!\ldots i_n!}X_1^{i_1}X_2^{i_2}\ldots X_n^{i_n}\end{aligned}$$

Let us take $X_i = 1$ for all n to get,

$$\sum\nolimits_{i_1+i_2+\ldots+i_n=k}\frac{(i_1+\ldots+i_n)!}{i_1!i_2!\ldots i_n!}=n^k,$$

a nice formula to my taste.

4. *Show that in any ring a prime element is irreducible.* (10 pts.)

Proof: Let R be any (commutative) ring (with 1). Recall that an element $p \in R \setminus R^*$ which is not a zero divisor is called **prime** if whenever p divides xy then p divides either x or y. An element $p \in R \setminus R^*$ which is not a zero divisor is called **irreducible** if whenever $p = xy$ then either x or y is in R^*. Assume p is prime in R. Assume $p = xy$. Then p divides xy. Since p is prime, this implies that p divides either x or y. The situation being symmetrical with respect to x and y, we may assume that p divides x. Let $z \in R$ be such that $x = pz$. Now $p = xy = pzy$ and $p(1 - zy) = 0$. Since p is not a zerodivisor, this implies that $1 - zy = 0$, i.e. $zy = 1$ and so $= 1$ and so $y \in R^*$.

5. *Let f_n be the number of words in letters a, b and c's of length n without the subword abc.*

5a. *Find a recursive formula for f_n.*

5b. *Compute f_6 and f_7.*

(20 pts.)

Answer: Clearly $f_1 = 1$ (the empty word), $f_2 = 9$, $f_3 = 27 - 1 = 26$ (all but abc), $f_4 = 3^4 - 6$ (all but $abca$, $abcb$, $abcc$, $aabc$, $babc$, $cabc$). Now let $n \geq 3$. Given a word w without abc of length $n - 1$, we can freely add a or b to the end of w to obtain the words wa and wb without

abc. We can also add c to get the words wa, wb and wc without abc in case the word w of length $n - 1$ does not end with ab. If g_n denotes the number of words without abc that end with ab then, the above discussion shows that

$$f_n = 3(f_{n-1} - g_{n-1}) + 2g_{n-1}.$$

So let us compute g_n. Clearly to any word w without abc of length $n - 2$, we can add ab to the end to get wab, a word without abc and that ends with ab. Thus,

$$g_n = f_{n-2}.$$

Therefore

$$f_n = 3(f_{n-1} - g_{n-1}) + 2g_{n-1} = 3(f_{n-1} - f_{n-3}) + 2f_{n-3} = 3f_{n-1} - f_{n-3}.$$

By using this formula we can compute f_n recursively:

$f_1 = 1$
$f_2 = 9$,
$f_3 = 3f_2 - f_0 = 27 - 1 = 26$
$f_4 = 3f_3 - f_1 = 3\times 26 - 1 = 75$
$f_5 = 3f_4 - f_2 = 3\times 75 - 9 = 216$
$f_6 = 3f_5 - f_3 = 3\times 216 - 26 = 622$
$f_7 = 3f_6 - f_4 = 3\times 622 - 75 = 1866 - 75 = 1791$.

6. *How many irreducible polynomials are there in* $\mathbb{Z}[X]$ *of the form* $X^2 + aX + b$ *where* a, b $\in \{-2, -1, 0, 1, 2\}$? (15 pts.)

Answer: A reducible polynomial of the form $X^2 + aX + b$ must be a product of two monic polynomials of degree 1, thus they must have at least one root in $\mathbb{Z}$. Since the roots are given by

$$\frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

the coefficients a and b must satisfy the following two conditions:

a) the discriminant $a^2 - 4b$ must be a perfect square in $\mathbb{Z}$, and
b) Since an eventual root must be in $\mathbb{Z}$ and not in $\mathbb{Q}$, $-a + \sqrt{(a^2 - 4b)}$ must be divisible by 2, i.e. $a^2 - 4b$ and a must be of the same parity, but this is always the case.

We compute $a^2 - 4b$ case by case to see which pairs (a, b) satisfy the condition a (condition b is automatically satisfied):

$a^2 - 4b$	$a = -2$	$a = -1$	$a = 0$	$a = 1$	$a = 2$
$b = -2$	**12**	9	**8**	9	**12**
$b = -1$	**8**	**5**	4	**5**	**8**
$b = 0$	4	1	0	1	4
$b = 1$	0	**–3**	**–4**	**–3**	0
$b = 2$	**–4**	**–7**	**–8**	**–7**	**–4**

We printed bold face the output $a^2 - 4b$ in case it is not a square. There are 15 of them.
So there are 15 irreducible polynomials that satisfy the given conditions.

7. *Find all irreducible polynomials of degree* 3 *of* $(\mathbb{Z}/2\mathbb{Z})[X]$. (15 pts.)

Answer: Clearly a reducible polynomial of degree 3 must have a factor of degree 1, i.e. must be divisible either by X or by $X - 1$, hence it must have a root (either 0 or 1). Let us list all polynomials of degree 3 and find out the ones that do not have a root, these are the irreducible ones:

Polynomial $f(X)$	$f(0)$	$f(1)$	Result	Decomposition
X^3	0	1	reducible	XXX
$X^3 + 1$	1	0	reducible	$(X + 1)(X^2 + X + 1)$
$X^3 + X$	0	0	reducible	$X(X + 1)^2$
$X^3 + X + 1$	1	1	**irreducible**	
$X^3 + X^2$	0	0	reducible	$X^2(X + 1)$
$X^3 + X^2 + 1$	1	1	**irreducible**	
$X^3 + X^2 + X$	0	1	reducible	$X(X^2 + X + 1)$
$X^3 + X^2 + X + 1$	1	0	reducible	$(X+1)^3$