Math 111 Complex Numbers

Ali Nesin

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Part I.

1. Show that if $ax^2 + bx + c = 0$ and $a \neq 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (Here, you can treat a, b, c and x as real numbers.)
2. Show that for any complex number α there is a complex number β such that $\beta^2 = \alpha$.
3. Let α, β, γ be three complex numbers with $\alpha \neq 0$. Let δ be a complex number such that $\delta^2 = \beta^2 - 4\alpha\gamma$. Show that if $x = (-\beta \pm \delta)/2\alpha$ then $\alpha x^2 + \beta x + \gamma = 0$.
4. Conclude from the questions above that any polynomial $\alpha X^2 + \beta X + \gamma$ with α, β, $\delta \in \mathbb{C}$ has a root in $\mathbb{C}$.
5. Let $a, b, c \in \mathbb{R}$. Suppose that $\alpha \in \mathbb{C}$ is a root of the polynomial $aX^2 + bX + c$. Find the other root of $aX^2 + bX + c$.
6. Let $n \in \mathbb{N} \setminus \{0\}$ and $\alpha \in \mathbb{C}$. Show that there is a $\beta \in \mathbb{C}$ such that $\beta^n = \alpha$.

Part II.

7. Show that $\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta$.
8. Find a formula for $\sin 4\theta$ in terms of $\cos \theta$ and $\sin \theta$.

Part III.

Let $\zeta = \cos 72^\circ + i \sin 72^\circ$ and $a = \zeta + \zeta^4$.

9. Show that $\zeta^5 = 1$.
10. Show that $1, \zeta, \zeta^2, \zeta^3, \zeta^4$ are all the roots of the polynomial $X^5 - 1$.
11. Draw $\zeta, \zeta^2, \zeta^3, \zeta^4$ on the complex plane, i.e. show their geometric representation on the plane $\mathbb{R}^2$.
12. Show that $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$.
13. Show that ζ^4 is the conjugate of ζ.
14. Conclude from above that $a = 2\cos 72^\circ$.
15. Show that $a^2 + a - 1 = 0$.
16. Find $\cos 72^\circ$.