Complex Analysis II

Midterm (Take Home, First Part)
April-May 2001
Ali Nesin

Before starting the exam, review the convergence of power series.
No books, no notes and no discussions are allowed after you start the exam.
There will be a more theoretical second part exam later on.

1. Describe the sets

1a. $A = \{z \in \mathbb{C} : \dfrac{|z-1|}{|z+1|} = 1\}$

1b. $B = \{z \in \mathbb{C} : |z|^2 = \mathrm{Im}(z)\}$.

1c. $C = \{z \in \mathbb{C} : |z^2 - 1| < 1\}$.

2. For this question do not use any known results.

2a. Let X and Y be a metric space and let $(f_n)_n$ be a set of continuous functions from X into Y.

Show that if the the sequence f_n converges uniformly to f then f is continuous.

2b. Let $D \subseteq \mathbb{C}$ be an open subset and let $f_i : D \to \mathbb{C}$ be continuous. Suppose $|f_i(z)| \le M_i$ for $z \in D$ and that $\sum_{i=0}^{\infty} M_i$ converges. Show that for each $z \in D$, $\sum_{i=0}^{\infty} f_i(z)$ converges and that the function $f : D \to \mathbb{C}$ defined by $f(z) = \sum_{i=0}^{\infty} f_i(z)$ is continuous on D. (Hint: Use 2a.)

2c. Show that the series $\sum_{i=0}^{\infty} iz^i$ converges uniformly for $|z| < \alpha < 1$ and is continuous in B(0, 1). (Hint: Use 2b.)

3. Discuss the convergence of the series ($z \in \mathbf{C}$):

3a. $\sum_{i=0}^{\infty} iz^i$

3b. $\sum_{i=1}^{\infty} z^i / i^2$

3c. $\sum_{i=0}^{\infty} (1 + (-1)^i)^i z^i$

3d. $\sum_{i=0}^{\infty} z^{i!}$

3e. $\sum_{i=0}^{\infty} (i + 2^i) z^i$

4. Suppose $\sum c_n z^n$ has radius of convergence R. Find the radius of convergence of

$\sum n^p c_n z^n$

$\sum |c_n| z^n$

$\sum c_n^2 z^n$

5. Which of the following polynomials are analytic?

$P(x + iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

$P(x + iy) = x^2 + iy^2$
$P(x + iy) = 2xy + i(y^2 - x^2)$

Express the analytic ones as a function of z and show that for these we have $P'(z) = P_x(z)$.

6. By using Cauchy-Riemann equations, show that a nonconstant analytic functions cannot take only imaginary values.

7. Show that $\sum_{i=1}^{\infty} z^n / n$ converges at all points of the unit circle except $z = 1$.

8. Suppose that $\sum a_k = A$ and $\sum b_k = B$ converge absolutely.

8a. Show that $\sum d_k$ where $d_k = \sum_{i=0}^{k} |a_i||b_{k-i}|$ converges.

8b. Let

$A_n = a_o + ... + a_n$
$B_n = b_o + ... + b_n$
$C_n = c_o + ... + c_n$

where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$. Show that $A_n B_n = C_n + R_n$ where $|R_n| \leq d_{n+1} + ... + d_{2n}$

8c. Show that $\sum c_k = AB$.

9. The Cauchy product of $\sum a_n z^n$ and $\sum b_n z^n$ is defined as $\sum c_n z^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Show that if $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2 respectively, then $\sum c_n z^n$ converges to $(\sum a_n z^n)(\sum b_n z^n)$ for $|z| < \min(R_1, R_2)$.

10. What is the Cauchy product of $\sum_{n=0}^{\infty} z^n$ with itself?