Before starting the exam, review the convergence of power series.
No books, no notes and no discussions are allowed after you start the exam.
There will be a more theoretical second part exam later on.

1. Describe the sets
   1a. \( A = \{ z \in \mathbb{C} : \frac{|z| - 1}{|z| + 1} = 1 \} \)
   1b. \( B = \{ z \in \mathbb{C} : |z|^2 = \text{Im}(z) \} \)
   1c. \( C = \{ z \in \mathbb{C} : |z^2 - 1| < 1 \} \)

2. For this question do not use any known results.
   2a. Let \( X \) and \( Y \) be a metric space and let \( (f_n) \) be a set of continuous functions from \( X \) into \( Y \).
       Show that if the sequence \( f_n \) converges uniformly to \( f \) then \( f \) is continuous.
   2b. Let \( D \subseteq \mathbb{C} \) be an open subset and let \( f_i : D \rightarrow \mathbb{C} \) be continuous. Suppose \( |f_i(z)| \leq M_i \) for \( z \in D \) and that \( \sum_{i=0}^{\infty} M_i \) converges. Show that for each \( z \in D \), \( \sum_{i=0}^{\infty} f_i(z) \) converges and that the function \( f : D \rightarrow \mathbb{C} \) defined by \( f(z) = \sum_{i=0}^{\infty} f_i(z) \) is continuous on \( D \). (Hint: Use 2a.)
   2c. Show that the series \( \sum_{i=0}^{\infty} i z^i \) converges uniformly for \( |z| < \alpha \) and is continuous in \( B(0, 1) \). (Hint: Use 2b.)

3. Discuss the convergence of the series (\( z \in \mathbb{C} \)):
   3a. \( \sum_{i=0}^{\infty} i z^i \)
   3b. \( \sum_{i=1}^{\infty} i z^i / i^2 \)
   3c. \( \sum_{i=1}^{\infty} (1 + (-1)^i) i z^i \)
   3d. \( \sum_{i=0}^{\infty} z^n \)
   3e. \( \sum_{i=0}^{\infty} (i + 2^i) z^i \)

4. Suppose \( \sum c_n z^n \) has radius of convergence \( R \). Find the radius of convergence of
   \( \sum n^p c_n z^n \)
   \( \sum f_n |z^n | \)
   \( \sum c_n^2 z^n \)

5. Which of the following polynomials are analytic?
   \( P(x + iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y) \)
\[ P(x + iy) = x^2 + iy^2 \]
\[ P(x + iy) = 2xy + i(y^2 - x^2) \]
Express the analytic ones as a function of \( z \) and show that for these we have \( P'(z) = P_x(z). \)

6. By using Cauchy-Riemann equations, show that a nonconstant analytic functions cannot take only imaginary values.

7. Show that \( \sum_{i=1}^{\infty} z^n / n \) converges at all points of the unit circle except \( z = 1 \).

8. Suppose that \( \sum a_k = A \) and \( \sum b_k = B \) converge absolutely.

8a. Show that \( \sum d_k \) where \( d_k = \sum_{i=0}^{k} |a_i||b_{k-i}| \) converges.

8b. Let
\[
A_n = a_0 + \ldots + a_n, \\
B_n = b_0 + \ldots + b_n, \\
C_n = c_0 + \ldots + c_n
\]
where \( c_k = \sum_{i=0}^{k} a_i b_{k-i} \). Show that \( A_nB_n = C_n + R_n \) where \( |R_n| \leq d_{n+1} + \ldots + d_{2n} \)

8c. Show that \( \sum c_k = AB. \)

9. The Cauchy product of \( \sum a_n z^n \) and \( \sum b_n z^n \) is defined as \( \sum c_n z^n \) where \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \). Show that if \( \sum a_n z^n \) and \( \sum b_n z^n \) have radii of convergence \( R_1 \) and \( R_2 \) respectively, then \( \sum c_n z^n \) converges to \( (\sum a_n z^n)(\sum b_n z^n) \) for \( |z| < \min(R_1, R_2) \).

10. What is the Cauchy product of \( \sum_{n=0}^{\infty} z^n \) with itself?