Complex (and Real) Analysis Final
Math 331
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1. Let $f(x) = x^3 - 3x + 5$. Show that $f(\ln a) = 6$ for some $a > 1$.

2. Find $\cos(15^\circ)$.

3. Express $\sin(4x)$ and $\cos(4x)$ in terms of $\sin x$ and $\cos x$. (Prove your formula).

4. Draw with as much care as possible the graph of

$$f(x) = \frac{x^2}{(x-1)(x+2)}$$

5. Show that $\lim_{x \to 5} (x^2 - 3x + 5) = 15$ by using the definition of limits.

6. By using the definition of continuity, show that the function $f(x) = \dfrac{x}{x-1}$ is continuous in its domain of definition.

7. Is the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

continuous at 0? (Justify your answer).

8. Prove that if $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$ then $\lim_{n \to \infty} a_n b_n = ab$.

9. Let $f_n(x) = \dfrac{1}{1+x^n}$.

9a. Find the set $A = \{x \in \mathbb{R} : (f_n(x))_n \text{ converges}\}$.
For $x \in A$, let $f(x) = \lim_{n \to \infty} f_n(x)$.
9b. What is f?
9c. Is the convergence uniform? Justify your answer.
9d. Discuss the uniform convergence of $(f_n)_n$ in the (open or closed) intervals contained in A.

10. Discuss the convergence and absolute convergence of the alternating series

$$1 - 1/2^\alpha + 1/3^\alpha - 1/4^\alpha + \ldots$$

for various values of α.

11. Given two continuous numerical functions f and g, show that $\max\{f(x), g(x)\}$ is also continuous.

12. Let f be a continuous numerical function on $[a, b]$. Let $x_1, \dots, x_n$ be arbitrary points in $[a, b]$. Show that $f(x_o) = \frac{1}{n}(f(x_1) + \dots + f(x_n))$ for some $x_o \in [a, b]$.

13a. What is the Taylor series of e^x?
13b. Estimate the error made in replacing the function e^x on the interval [0, 1] by its Taylor polynomial of degree 10.
13c. On what interval $[0, h]$ does the function e^x differ from its Taylor polynomial of degree 10 by no more than 10^{-7}?
13d. For what value of n does the function e^x differ from its Taylor polynomial of degree n by no more than 10^{-7} on the interval [0, 1]?