Complex Analysis

Resit
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1. Going back to the definitions, find all the complex numbers z where the function $f(z) = |z|^2$ is differentiable.

2. Prove the Cauchy-Riemann equations for an analytic function f.

3. Let $f = u + iv$ be an entire function where u and v are real valued functions on **C**. Find all functions $w : \mathbb{C} \to \mathbb{R}$ such that the function $u + iw$ is entire.

4. Is there a function $v : \mathbb{C} \to \mathbb{R}$ such that the function $f(x + iy) = \sin(x) + iv(x + iy)$ is entire? If so find it, otherwise disprove the existence.

5. Find the Laurent series of

$$f(z) = (z^2 - 1)/z$$

around $z_o = 0$.

6. Prove that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

7. Find the first four terms of the Laurent series of $f(z) = e^z/\sin z$ around 0.

8. Find $\int_{|z|=1/2} \frac{e^z - e^{-z}}{z^3} dz$.

9. Suppose f and g are analytic in a region D. Suppose also that $f(z)^2 = g(z)^2$ for $z \in D$. What can you say about f and g?

10. Suppose f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all z. What can you say about the relationship of f and g?

11. Suppose f is analytic in a domain Ω containing the unit disc and $|f(z)| > 2$ for all $|z| = 1$ and also $f(0) = 1$. Show that f must be equal to zero for some point in the unit disc.

12. Suppose f is any analytic function in the open unit disc. Show that there must be a sequence $(z_n)_n$ with $|z_n| \to 1$ such that $f(z_n)$ is bounded.