Complex Analysis

Final Exam
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Open book. (You don’t have to!) Justify your answers. (You have to!)

1. Find the Taylor series of
   \[ f(z) = z^3 + 2z - 1 \]
   \[ g(z) = 1/z \]
   around \( z_0 = 1 \).

   **Answer:** For \( f \) analytic around \( a \), we know that the Taylor series of \( f \) is given by
   \[ f(z) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(a)}{n!} (z-a)^n. \]
   We can apply this formula to find the coefficients of the Taylor series. For the first one this works well:
   \[
   \begin{align*}
   f^{(0)}(1) &= f(1) = 2, \\
   f^{(1)}(1) &= (3z^2 + 2)(z = 1) = 5 \\
   f^{(2)}(1) &= 6z(z = 1) = 6 \\
   f^{(3)}(1) &= 6 \\
   f^{(4)}(1) &= 0
   \end{align*}
   \]
   Thus \( f(z) = 2 + 5(z-1) + 3(z-1)^2 + (z-1)^3 \).
   One can check the equality \( z^3 + 2z - 1 = 2 + 5(z-1) + 3(z-1)^2 + (z-1)^3 \) to be sure that there is no mistake.
   For the second, a second method works better:
   \[
   g(z) = \frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n} (1-z)^n = \sum_{n} (-1)^n (z-1)^n
   \]

2. Find the first four terms of the Laurent series of \( f(z) = 1/\sin z \) around 0.

   **Answer:** Since \( \lim_{z \to 0} zf(z) = 1 \), \( f(z) \) has a pole of order 1 at 0. (At this point we can apply the Taylor series formula to the analytic function \( z/\sin z \). Hence the Laurent series of \( f(z) \) around 0 is given by \( f(z) = a_{-1}/z + a_0 + a_1z + a_2z^2 + ... \) Clearly \( a_{-1} = \lim_{z \to 0} zf(z) = \lim_{z \to 0} z/\sin z = 1 \). Thus \( f(z) = 1/z + a_0 + a_1z + a_2z^2 + ... \) (At this point we can apply the Taylor series formula to the function \( f(z) - 1/z \). Thus \( a_0 = \lim_{z \to 0} (f(z) - 1/z) = \lim_{z \to 0} (1/\sin z - 1/z) = \lim_{z \to 0} (z - \sin z)/z\sin z = \lim_{z \to 0} (1 - \cos z)/(\sin z + z\cos z) = \lim_{z \to 0} \sin z/(2\cos z - \sin z) = 0 \) by applying L’Hospital’s Rule twice. Hence \( a_0 = 0 \) and \( f(z) = 1/z + a_1z + a_2z^2 + ... \). Hence \( a_1 = \lim_{z \to 0} (f(z) - 1/z)/z = \lim_{z \to 0} (z - \sin z)/z^2\sin z = \lim_{z \to 0} (1 - \cos z)/(2\sin z + z^2\cos z) = \lim_{z \to 0} \sin z/(2\sin z + 4z\cos z - z^2\sin z) = \lim_{z \to 0} \cos z/(6\cos z - 6z\sin z - z^2\cos z) = 1/6 \). Thus \( f(z) = 1/z + z/6 + a_2z^2 + ... \) and \( a_2 = \lim_{z \to 0} (f(z) - 1/z - z/6)/z^2 \). Applying L’Hospital’s Rule as many times at it is needed, one easily finds \( a_2 \). (This short answer was enough to get the highest score).
3. Find \( \int_{|z|=1} \frac{e^z - e^{-z}}{z^4} \, dz \).

**Answer:** Apply one of the versions of Cauchy’s Integral Formula that states the following: Let \( U \) be an open set and \( f \) analytic in \( U \). If \( f \) is a closed rectifiable curve in \( U \) such that \( n(\gamma, w) = 0 \) for all \( w \in \mathbb{C} \setminus U \) then, for all \( a \in U \setminus \{\gamma\} \),

\[
 f^{(k)}(a)n(\gamma, a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} \, dz .
\]

Take \( U \) to be the plane \( \mathbb{C} \), \( f \) to be \( e^z - e^{-z} \), \( \gamma \) to be the unit circle positively oriented, \( a = 0 \) and \( k = 3 \). The hypothesis are satisfied. Thus the integral is equal to

\[
 2\pi i f^{(3)}(0)n(\gamma, 0)/3! = 2\pi i f^{(3)}(0)/3! = 2\pi i \times 2/3! = 2\pi i/3 .
\]

**Another Solution:** Since \( e^z - e^{-z} = 2z + 2z^3/3! + 2z^5/5! \) (Taylor series around 0), \( e^z - e^{-z} = 2z^3 + 2/3! - 2z^5/5! \) (Taylor series around 0). Thus \((e^z - e^{-z})/z^4 = 2/3! + 2/3!z + 2/5!z^5 + \ldots \) (Laurent series around 0). Now apply the theorem about the Laurent series that states the following: Let \( f \) be analytic in the open annulus \( \text{ann}(a; R_1, R_2) \). Then

\[
 f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z-a)^n
\]

and the coefficients are given by

\[
 a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} \, dz
\]

where \( \gamma \) is any circle inside the annulus. Take \( a = 0 \), \( R_1 = 2 \), \( R_2 = 1/2 \),

\[
 f(z) = (e^z - e^{-z})/z^4 , \quad n = -1 , \quad \gamma \text{ the unit circle. Since } a_{-1} = 2/3! = 1/3 , \text{ we get the same answer, namely } 2\pi i/3 .
\]

**Another Solution:** We apply the Residue Theorem to \( f(z) = (e^z - e^{-z})/z^4 \). Recall the Residue Theorem: If \( f \) is analytic in the connected open set \( U \) except for the isolated singularities at \( a_1, \ldots, a_n \), if \( \gamma \) is a closed rectifiable curve in \( U \) which does not pass through any of the points \( a_k \) and if \( \gamma \cap U = \emptyset \) then

\[
 \frac{1}{2\pi i} \int_{\gamma} f = \sum_{k=1}^{n} \text{Res}(f; a_k) .
\]

Take \( U \) to be the unit disk, \( f(z) = (e^z - e^{-z})/z^4 , \quad n = 1 , \quad a_1 = 0 , \quad \gamma \text{ the unit circle oriented positively. The hypothesis are satisfied. Since } \text{Res}(f; 0) = a_{-1} = 2/3! = 1/3 \text{ and } n(\gamma; 0) = 1 , \text{ we again obtain } 2\pi i/3 .
\]

4. Suppose \( f \) is analytic in a simply connected region \( \Omega \), \( \gamma \) is a circle in \( \Omega \) and \( f \) has no zeroes on \( \gamma \). In terms of the zeroes of \( f \) inside \( \gamma \) what is the value of

\[
 \int_{\gamma} \frac{f'}{f} \, z^p \, dz ?
\]

**Answer:** We apply the generalization of the Argument Principle: Let \( f \) be meromorphic in the connected open set \( \Omega \) with zeroes \( z_1, \ldots, z_n \) and poles \( p_1, \ldots, p_m \) counted according to multiplicity. If \( g \) is analytic in \( \Omega \) and \( \gamma \) is any closed rectifiable curve in \( \Omega \) with \( \gamma = 0 \) and not passing through any \( z_k \) or \( p_k \), then

\[
 \frac{1}{2\pi i} \int_{\gamma} g \frac{f'}{f} = \sum_{k=1}^{n} g(z_k)n(\gamma; z_k) - \sum_{k=1}^{m} g(p_k)n(\gamma; p_k) .
\]

We take \( g(z) = z^p \). Since \( f \) is analytic in \( \Omega \), \( f \) has no poles in \( \Omega \). Let \( z_1, \ldots, z_n \) be the zeroes of \( f \) in \( \Omega \) counted according to multiplicity. Then the theorem says

\[
 \int_{\gamma} \frac{f'}{f} \, z^p \, dz = 2\pi i \sum_{k=1}^{n} z_k^p .
\]

5. Suppose that \( \Omega \) is a region containing the disc \( D \), \( f \) is a nonconstant analytic function in \( \Omega \) such that \( |f| \) is constant on \( \partial D \). Show that \( f \) has at least one zero inside \( D \).

Note that \( f \) cannot be 0 on \( \partial D \), because otherwise the zeroes of \( f \) would have an accumulation point and \( f \) would be zero, hence a constant. Assume \( f \) has no zero in \( D \) and consider \( g(z) = 1/f(z) \) defined on \( D \). Then \( g \) is analytic in \( D \).
6. Suppose $f$ and $g$ are analytic in a region $D$. Suppose also that $f(z)^2 = g(z)^2$ for $z \in D$. What can you say about $f$ and $g$?

We will show that $f = g$ or $f = -g$ on $D$. Since $D$ is connected, it is enough to show that this is so on any open and bounded disk of $D$. Hence we may assume that $D$ is an open and bounded disk. If $f = 0$ on $D$ we are done. Otherwise, $D$ being bounded, $f$ has finitely many zeroes in $D$. The set $E = D \setminus \{ z : f(z) = 0 \}$ is still connected. Now $E_+ = \{ z \in E : f(z) = g(z) \}$ and $E_- = \{ z \in E : f(z) = -g(z) \}$ disconnect $E$. Hence $E = E_+$ or $E = E_-.$

7. Suppose $f$ and $g$ are entire functions such that $|f(z)| \leq |g(z)|$ for all $z$. What can you say about the relationship of $f$ and $g$?

8. Suppose $f$ is analytic in a domain $\Omega$ containing the unit disc and $|f(z)| > 2$ for all $|z| = 1$ and also $f(0) = 1$. Show that $f$ must be equal to zero for some point in the unit disc.

9. Suppose $f$ is any analytic function in the open unit disc. Show that there must be a sequence $(z_n)_n$ with $|z_n| \to 1$ such that $f(z_n)$ is bounded.

10. Suppose $P(z, w)$ is a polynomial in two complex variables. Suppose $w_0$ is such that $P(z, w_0)$ has only simple zeroes (as a polynomial in one variable). Show that the same property holds for all $w$ near $w_0$. **Hint:** Recall the following weaker version of Rouché’s Theorem: Suppose $f$ and $g$ are meromorphic in a neighborhood of $\overline{B}(a; R)$ with no zeroes or poles on the circle $C(a; R)$. If $Z_f$, $Z_g$, $P_f$ and $P_g$ denote the number of zeroes and poles of $f$ and $g$ inside $\gamma$ counted according to their multiplicities and if $|f(z) + g(z)| < |f(z)|$ on $\gamma$, then $Z_f - P_f = Z_g - P_g$. Also use the fact that the polynomial function $P(z, w)$ is uniformly continuous on any compact subset of $\mathbb{C}^2$. 