Complex Analysis

Final Exam
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Open book. (You don't have to!)
Justify your answers. (You have to!)

1. *Find the Taylor series of*

$f(z) = z^3 + 2z - 1$
$g(z) = 1/z$

around $z_o = 1$.

Answer: For f analytic around a, we know that the Taylor series of f is given by $f(z) = \sum_{n\in N} \frac{f^{(n)}(a)}{n!}(z-a)^n$. We can apply this formula to find the coefficients of the Taylor series. For the first one this works well:

$f^{(0)}(1) = f(1) = 2,$
$f^{(1)}(1) = (3z^2 + 2)(z = 1) = 5$
$f^{(2)}(1) = 6z(z = 1) = 6$
$f^{(3)}(1) = 6$
$f^{(4)}(z) = 0$

Thus $f(z) = 2 + 5(z-1) + 3(z-1)^2 + (z-1)^3$.

One can check the equality $z^3 + 2z - 1 = 2 + 5(z-1) + 3(z-1)^2 + (z-1)^3$ to be sure that there is no mistake.

For the second, a second method works better:

$$g(z) = \frac{1}{z} = \frac{1}{1-(1-z)} = \sum_n (1-z)^n = \sum_n (-1)^n (z-1)^n$$

2. *Find the first four terms of the Laurent series of* $f(z) = 1/\sin z$ *around* 0.

Answer: Since $\lim_{z\to 0} zf(z) = 1$, $f(z)$ has a pole of order 1 at 0. (At this point we can apply the Taylor series formula to the analytic function $z/\sin z$). Hence the Laurent series of $f(z)$ around 0 is given by $f(z) = a_{-1}/z + a_0 + a_1z + a_2z^2 + \ldots$ Clearly $a_{-1} = \lim_{z\to 0} zf(z) = \lim_{z\to 0} z/\sin z = 1$. Thus $f(z) = 1/z + a_0 + a_1z + a_2z^2 + \ldots$ (At this point we can apply the Taylor series formula to the function $f(z) - 1/z$). Thus $a_0 = \lim_{z\to 0} (f(z) - 1/z) = \lim_{z\to 0} (1/\sin z - 1/z) = \lim_{z\to 0} (z - \sin z)/z\sin z = \lim_{z\to 0} (1 - \cos z)/(\sin z + z\cos z) = \lim_{z\to 0} \sin z/(2\cos z - z\sin z) = 0$ by applying L'Hospital's Rule twice. Hence $a_0 = 0$ and $f(z) = 1/z + a_1z + a_2z^2 + \ldots$. Hence $a_1 = \lim_{z\to 0} (f(z) - 1/z)/z = \lim_{z\to 0} (z - \sin z)/z^2\sin z = \lim_{z\to 0} (1 - \cos z)/(2z\sin z + z^2\cos z) = \lim_{z\to 0} \sin z/(2\sin z + 4z\cos z - z^2\sin z) = \lim_{z\to 0} \cos z/(6\cos z - 6z\sin z - z^2\cos z) = 1/6$. Thus $f(z) = 1/z + z/6 + a_2z^2 + \ldots$ and $a_2 = \lim_{z\to 0} (f(z) - 1/z - z/6)/z^2$. Applying L'Hospital's Rule as many times at it is needed, one easily finds a_2. (This short answer was enough to get the highest score).

3. Find $\int_{|z|=1} \frac{e^z - e^{-z}}{z^4} dz$.

Answer: Apply one of the versions of Cauchy's Integral Formula that states the following: *Let U be an open set and f analytic in U. If γ is a closed rectifiable curve in U such that n(γ; w) = 0 for all w ∈* **C** *\ U then, for all a ∈ U \ {γ},* $f^{(k)}(a)n(\gamma, a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}dz$. Take U to be the plane **C**, f to be $e^z - e^{-z}$, γ to be the unit circle positively oriented, $a = 0$ and $k = 3$. The hypothesis are satisfied. Thus the integral is equal to $2\pi i f^{(k)}(a)n(\gamma, a)/k! = 2\pi i f^{(3)}(0)n(\gamma, 0)/3! = 2\pi i f^{(3)}(0)/3! = 2\pi i \times 2/3! = 2\pi i/3$.

Another Solution: Since $e^z = 1 + z + z^2/2! + z^3/3! + \ldots$ and $e^{-z} = 1 - z + z^2/2! - z^3/3! + \ldots$ (Taylor series around 0), $e^z - e^{-z} = 2z + 2z^3/3! + 2z^5/5!$ (Taylor series around 0). Thus $(e^z - e^{-z})/z^4 = 2/z^3 + 2/3!z + 2z/5! + \ldots$ (Laurent series around 0). Now apply the theorem about the Laurent series that states the following: *Let f be analytic in the open annulus* ann(a; R_1, R_2). *Then* $f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z-a)^n$ *and the coefficients are given by* $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz$ *where γ is any circle inside the annulus*. Take $a = 0$, $R_1 = 2$, $R_2 = 1/2$, $f(z) = (e^z - e^{-z})/z^4$, $n = -1$, γ the unit circle. Since $a_{-1} = 2/3! = 1/3$, we get the same answer, namely $2\pi i/3$.

Another Solution: We apply the Residue Theorem to $f(z) = (e^z - e^{-z})/z^4$. Recall the Residue Theorem: *If f is analytic in the connected open set U except for the isolated singularities at* $a_1, \ldots, a_n$, *if γ is a closed rectifiable curve in U which does not pass through any of the points* a_k *and if γ ≈ 0 in U then* $\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} n(\gamma; a_k)\mathrm{Res}(f; a_k)$. Take U to be the unit disk, $f(z) = (e^z - e^{-z})/z^4$, $n = 1$, $a_1 = 0$, γ the unit circle oriented positively. The hypothesis are satisfied. Since Res(f; 0) = $a_{-1} = 2/3! = 1/3$ and $n(\gamma; 0) = 1$, we again obtain $2\pi i/3$.

4. Suppose f is analytic in a simply connected region Ω, γ is a circle in Ω and f has no zeroes on γ. In terms of the zeroes of f inside γ what is the value of $\int_\gamma \frac{f'}{f} z^p dz$?

Answer: We apply the generalization of the Argument Principle: *Let f be meromorphic in the connected open set Ω with zeroes* $z_1, \ldots, z_n$ *and poles* $p_1, \ldots, p_m$ *counted according to multiplicity. If g is analytic in Ω and γ is any closed rectifiable curve in Ω with γ ≈ 0 and not passing through any* z_k *or* p_k, *then* $\frac{1}{2\pi i}\int_\gamma g\frac{f'}{f} = \sum_{k=1}^{n} g(z_k)n(\gamma; z_k) - \sum_{i=1}^{n} g(p_k)n(\gamma; p_k)$. We take $g(z) = z^p$. Since f is analytic in Ω, f has no poles in Ω. Let $z_1, \ldots, z_n$ be the zeroes of f in Ω counted according to multiplicity. Then the theorem says $\int_\gamma \frac{f'}{f} z^p dz = 2\pi i \sum_{k=1}^{n} z_i^p$.

5. *Suppose that Ω is a region containing the disc D, f is a nonconstant analytic function in Ω such that |f| is constant on ∂D. Show that f has at least one zero inside D.*

Note that f cannot be 0 on ∂D, because otherwise the zeroes of f would have an accumulation point and f would be zero, hence a constant. Assume f has no zero in D and consider $g(z) = 1/f(z)$ defined on D. Then g is analytic in D

6. *Suppose f and g are analytic in a region D. Suppose also that* $f(z)^2 = g(z)^2$ *for* $z \in D$. *What can you say about f and g*?

We will show that $f = g$ or $f = -g$ on D. Since D is connected, it is enough to show that this is so on any open and bounded disk of D. Hence we may assume that D is an open and bounded disk. If $f = 0$ on D we are done. Otherwise, D being bounded, f has finitely many zeroes in D. The set $E = D \setminus \{z : f(z) = 0\}$ is still connected. Now $E_+ = \{z \in E : f(z) = g(z)\}$ and $E_- = \{z \in E : f(z) = -g(z)\}$ disconnect E. Hence $E = E_+$ or $E = E_-$.

7. Suppose f and g are entire functions such that $|f(z)| \le |g(z)|$ for all z. What can you say about the relationship of f and g?

8. Suppose f is analytic in a domain Ω containing the unit disc and $|f(z)| > 2$ for all $|z| = 1$ and also $f(0) = 1$. Show that f must be equal to zero for some point in the unit disc.

9. Suppose f is any analytic function in the open unit disc. Show that there must be a sequence $(z_n)_n$ with $|z_n| \to 1$ such that $f(z_n)$ is bounded.

10. Suppose $P(z, w)$ is a polynomial in two complex variables. Suppose w_o is such that $P(z, w_o)$ has only simple zeroes (as a polynomial in one variable). Show that the same property holds for all w near w_o. **Hint:** Recall the following weaker version of Rouché's Theorem: *Suppose f and g are meromorphic in a neighborhood of* $\overline{B}(a;R)$ *with no zeroes or poles on the circle* $C(a; R)$. *If* Z_f, Z_g, P_f *and* P_g *denote the number of zeroes and poles of f and g inside* γ *counted according to their multiplicities and if* $|f(z) + g(z)| < |f(z)|$ *on* γ, *then* $Z_f - P_f = Z_g - P_g$. Also use the fact that the polynomial function $P(z, w)$ is uniformly continuous on any compact subset of $\mathbf{C}^2$.