Final Exam on Math 114 (Set Theory)
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Notes: The questions are not so difficult as the number of points attached to them shows. Please read the text, understand the concepts and then read the questions attentively. Explain yourself clearly with short and correct English sentences. At most one idea per sentence! Use punctuation marks when necessary. Use paragraphs. Make sure that you state correctly what you know and what you are looking for. Draw pictures of sets, it will be helpful. Do not use logical symbols such as $\Rightarrow$, &, $\forall$.

A ***metric space*** is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function such that for all $x, y, z \in X$,

M1. $d(x, y) \geq 0$,
M2. $d(x, y) = 0$ if and only if $x = y$,
M3. $d(x, y) = d(y, x)$,
M4. $d(x, y) \leq d(x, z) + d(z, y)$.

Throughout the exam, we let (X, d) be a metric space.

1. Show that $|d(x, z)| \geq |d(x, y) - d(y, z)|$ for all $x, y, z \in X$. (3 pts.)

The ***open ball of center*** $a \in X$ ***and radius*** $r \in \mathbb{R}$ is the set $B(a, r) = \{x \in X : d(a, x) < r\}$. A union of a set of open balls is called an ***open subset***.

2. Show that a subset U of X is open iff for any $a \in U$ there is an open ball B such that $a \in B \subseteq U$. (2 pts.)

3. Show that a subset U of X is open iff for any $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$. (4 pts.)

4. Show that for any $a \in X$, $\bigcap_{n=1}^{\infty} B(a, 1/n) = \{a\}$. (3 pts.)

5. For $a \in X$ and $r \in \mathbb{R}$, the subset $\underline{B}(a, r) = \{x \in X : d(a, x) \leq r\}$ of X is called a ***closed ball***.

5a. Show that the complement of a closed ball is an open subset. (3 pts.)

5b. Show that for any $a \in X$, $\bigcap_{n=1}^{\infty} \underline{B}(a, 1/n) = \{a\}$. (3 pts.)

5c. Let $K \subseteq X$ be a subset and $a \notin K$. For $n = 1, 2, \ldots$ let $U_n = \underline{B}(a, 1/n)^c$. Show that $K \subseteq \bigcup_{n=1}^{\infty} U_n$. (3 pts.)

6. Show that
6a. $\varnothing$ and X are open sets. (2 pts.)
6b. The union of a set of open sets is open. (2 pts.)
6c. The intersection of finitely many open sets is open. (4 pts.)

7. Show that if $x, y \in X$ are two distinct points then there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (3 pts.)

8. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Then it is easy to show that $(\mathbb{R}, d)$ is a metric space. This metric on $\mathbb{R}$ is called the ***Euclidean*** or the ***usual metric*** on $\mathbb{R}$. When we will speak of $\mathbb{R}$ as a metric space, we will always have this metric in mind.

8a. Show that the interval $(0, 1]$ of $\mathbb{R}$ is not open. (3 pts.)
8b. Show that $[0, 1]^c$ is open. (2 pt.)

Let U be an open subset $\mathbb{R}$. For $x, y \in U$, set $x \equiv y$ iff $[x, y] \cup [y, x] \subseteq U$.
8c. Show that $\equiv$ is an equivalence relation on U. (3 pts.)
8d. Show that each equivalence is an open interval. (7 pts.)
8e. Conclude that U is a union of disjoint open intervals. (1 pt.)
8f. Conclude that U is a union of countably many disjoint open intervals. (7 pt.)

9. A subset B of the metric space X is ***bounded*** if $B \subseteq B(a, r)$ for some $a \in A$ and some $r \in \mathbb{R}$. Show that B is bounded if and only if for any $a \in \mathbb{R}$ there is an $r \in \mathbb{R}$ such that $B \subseteq B(a, r)$. (5 pts.)

10. Let K be a subset of the metric space X. A family $(U_i)_{i\in I}$ of open subsets of X is called an ***open cover*** of K if $K \subseteq \cup_{i\in I} U_i$. If $J \subseteq I$ and $K \subseteq \cup_{i\in J} U_i$, then we say that $(U_i)_{i\in J}$ is a ***subcover*** of the cover $(U_i)_{i\in I}$. A subset K of the metric space X is called ***compact*** if <u>any</u> open cover of K has a finite subcover.
10a. Show that a finite subset of X is compact. (2 pts.)
10b. Show that the union of finitely many compact subsets is compact. (3 pts.)
10c. Show that a compact subset of a metric space is bounded (see #7). (5 pts.)
10d. Let $K \subseteq X$ be a compact subset, $a \notin K$ and for $n = 1, 2, \ldots$ Show that $\underline{B}(a, 1/n) \cap K = \emptyset$ for some $n \in \mathbb{N}$. **Hint:** See 5a and 5c. (5 pts.)
10e. Conclude that the complement of a compact subset of X is open. (5 pts.)

11e. Show that a closed interval $[a, b]$ is a compact subset of $\mathbb{R}$ (with the usual metric). (20 pts.)