Math 152
MT on Continuity
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Let $a \in X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ a function. f is called ***continuous at*** a if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$, we have $|f(x) - f(a)| < \varepsilon$.

1. Let $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Assume that there is a $\alpha > 0$ such that $(a - \alpha, a + \alpha) \cap X = \{a\}$. Show that any function $f : X \to \mathbb{R}$ is continuous at a. (3 pts.)

2. Let X be a finite subset of $\mathbb{R}$. Show that any function $f : X \to \mathbb{R}$ is continuous at any point of X. (3 pts.)

3. Let $X = \{1/n : n = 1, 2, ...\}$. Find all continuous functions from X into $\mathbb{R}$. (5 pts.)

4. Show that the function defined by

$$f(x) = \frac{1 - x^2}{3 - 2x + x^3}$$

is continuous at any point $a \in \mathbb{R}$ where $3 - 2a + a^3 \neq 0$. (10 pts.)

5. Show that the function exp is continuous everywhere. (15 pts.)

6. Let $a \in X \subseteq \mathbb{R}$ and $f, g : X \to \mathbb{R}$ be continuous at a. Show that $f + g$ and fg is continuous at a. (3 + 4 pts.)

7. Let $a \in X$, $Y \subseteq \mathbb{R}$, $f : X \to Y$ be continuous at a and $g : Y \to \mathbb{R}$ be continuous at $f(a)$. Show that $g \circ f$ is continuous at a. (10 pts.)

8. Let $a \in X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ be continuous at a. Show that there is a $\alpha > 0$ such that f is bounded on $[a - \alpha, a + \alpha]$. (8 pts.)

9. Let $a \in X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R} \setminus \{0\}$ be continuous at a. Show that $1/f$ is continuous at a. (7 pts.)

10. Show that the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous at 0. (7 pts.)

11. Let $a \in X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. Show that f is continuous at a if and only if for any open interval I containing $f(a)$ there is an open interval J containing a such that $J \cap X \subseteq f^{-1}(I)$. (10 pts.)

12. Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. Show that f is continuous everywhere if and only if for any union U of open intervals, $f^{-1}(U)$ is also a union of open intervals. (15 pts.)