Math 151 / Math 113 Analysis

Midterm
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1. *Let D be the set of all real sequences. On D define the following binary relation*:

$$(x_n)_n \equiv (y_n)_n \Leftrightarrow (x_n - y_n)_n \text{ is Cauchy.}$$

Show that $\equiv$ *is an equivalence relation on D*. (15 pts.)

Proof: Let $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ be three sequences.

a) **Reflexivity:** We will show that the sequence $(x_n - x_n)_n$ is a Cauchy sequence. This is the constant 0 sequence. Let $c_n = c$ for all n. We will show that the constant sequence $(c_n)_n$ is Cauchy. Let $\varepsilon > 0$ be any real number. Let $N = 0$ (for example). Then for all $n, m > N$, $|c_n - c_m| = |c - c| = 0 < \varepsilon$.

b) **Symmetry:** Suppose that $(x_n)_n \equiv (y_n)_n$, i.e. that $(x_n - y_n)_n$ is Cauchy. We need to show that $(y_n)_n \equiv (x_n)_n$, i.e. that the sequence $(y_n - x_n)_n$ is Cauchy. This is the same as proving that the sequence $(-c_n)_n$ is Cauchy assuming that sequence $(c_n)_n$ is Cauchy. (To see this set $c_n = x_n - y_n$). Let $\varepsilon > 0$. Since $(c_n)_n$ is Cauchy there is an N such that for all $n, m > N$, $|c_n - c_m| < \varepsilon$. So for $n, m > N$,

$$|(-c_n) - (-c_m)| = |c_m - c_n| = |c_n - c_m| < \varepsilon.$$

This shows that the sequence $(-c_n)_n$ is Cauchy.

c) **Transitivity:** Suppose that $(x_n)_n \equiv (y_n)_n$ and $(y_n)_n \equiv (z_n)_n$. Thus the sequences $(x_n - y_n)_n$ and $(y_n - z_n)_n$ are Cauchy. We need to show that the sequence $(x_n - z_n)_n$ is Cauchy. Setting $c_n = x_n - y_n$ and $d_n = y_n - z_n$, we see that we have to show that the sequence $(c_n + d_n)_n$ is Cauchy assuming that the sequences $(c_n)_n$ and $(d_n)_n$ are Cauchy. Let $\varepsilon > 0$. Since the sequence $(c_n)_n$ is Cauchy there is an N_1 such that for all $n, m > N_1$, $|c_n - c_m| < \varepsilon/2$. Similarly, since the sequence $(d_n)_n$ is Cauchy there is an N_2 such that for all $n, m > N_2$, $|d_n - d_m| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for all $n, m > N$,

$$|(c_n + d_n) - (c_m + d_m)| = |(c_n - c_m) + (d_n - d_m)| \le |c_n - c_m| + |d_n - d_m|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This is what we needed to prove.

2. *Let* $(x_n)_n$ *be a sequence. Suppose that the sequence*

$$(|x_0| + |x_1| + \ldots + |x_n|)_n$$

is convergent. Show that the sequence $(x_0 + x_1 + \ldots + x_n)_n$ *is Cauchy*. (15 pts.)

Proof: Since the sequence $(|x_0| + |x_1| + \ldots + |x_n|)_n$ is convergent, it is Cauchy. Therefore there is an N such that for all $n > m > N$,

$$|(|x_0| + |x_1| + \ldots + |x_n|) - (|x_0| + |x_1| + \ldots + |x_m|)| < \varepsilon,$$

i.e.

$$||x_{m+1}| + \ldots + |x_n|| < \varepsilon,$$

i.e.

$$|x_{m+1}| + \ldots + |x_n| < \varepsilon.$$

Now for $n > m > N$ we have,

$$|(x_0 + x_1 + \ldots + x_n) - (x_0 + x_1 + \ldots + x_m)| = |x_{m+1} + \ldots + x_n| \le |x_{m+1}| + \ldots + |x_n| < \varepsilon.$$

This proves that the sequence $(x_0 + x_1 + \ldots + x_n)_n$ is Cauchy.

3. *Show that* $\lim_{n\to\infty} 1/r^n = 0$ if $r > 1$. (15 pts.)

Proof: Write $r = 1 + s$ with $s > 0$. Then for all n, $r^n = (1 + s)^n \geq 1 + ns$ (either by induction or by binomial expansion). By Archimedes Property, there is an N such that $1 < Ns$. Then for all $n > N$ we have, $|1/r^n - 0| = 1/r^n = 1/(1+s)^n \leq 1/(1 + ns) < 1/ns < 1/Ns < \varepsilon$. This proves that $\lim_{n\to\infty} 1/r^n = 0$.

4. *Show that if* $(x_n)_n$ *is a nonnegative bounded sequence and* $r > 1$ *then sequence*

$$(x_0 + x_1r^{-1} + ... + x_nr^{-n})_n$$

is Cauchy. (15 pts.)

Proof: Let $\varepsilon > 0$. Let B be such that $x_n < B$ for all n. Since the sequence $(r^{-n})_n$ converges to 0 by the previous question, there is an N such that $r^{-N} < \varepsilon(1 - 1/r)/B$. Then for all $n > m > N$, we have

$$|(x_0 + x_1r^{-1} + ... + x_nr^{-n}) - (x_0 + x_1r^{-1} + ... + x_mr^{-m})| = |x_{m+1}r^{-m-1} +... + x_nr^{-n}|$$
$$= x_{m+1}r^{-m-1} +... + x_nr^{-n} \leq Br^{-m-1} +... + Br^{-n} = B(r^{-m-1} +... + r^{-n})$$
$$= Br^{-m-1}(1 +... + r^{-n+m+1}) = Br^{-m-1}(1 - r^{-n+m+2})/(1 - r)$$
$$\leq Br^{-m-1}/(1 - r) = Br^{-N}/(1 - 1/r) < \varepsilon.$$

This is what we needed.

5. *Let* $(x_n)_n$ *be a sequence. Show that if the sequence* $(x_0 + x_1 + ... + x_n)_n$ *converges then* $\lim_{n\to\infty} x_n = 0$. (15 pts.)

First Proof: Let $y_n = x_0 + x_1 + ... + x_n$ and $\lim_{n\to\infty} y_n = a$. Then $x_n = y_n - y_{n-1}$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (y_n - y_{n-1}) = \lim_{n\to\infty} y_n - \lim_{n\to\infty} y_{n-1} = a - a = 0.$$

Second Proof: Let $\varepsilon > 0$. Since the sequence $(x_0 + x_1 + ... + x_n)_n$ converges, it is Cauchy. Therefore there is an N_1 such that for all $n > m > N_1$,

$$|x_{m+1} +... + x_n| = |(x_0 + x_1 + ... + x_n) - (x_0 + x_1 + ... + x_m)| < \varepsilon.$$

Let $N = N_1 + 1$. Then for all $n > N_1$, taking $m = n-1 > N_1 - 1 = N$ above, we see that $|x_n| < \varepsilon$. Thus $\lim_{n\to\infty} x_n = 0$.

6. *Let* $(x_n)_n$ *be a sequence such that for some* $0 < r < 1$,

$$|x_{n+2} - x_{n+1}| \leq r\,|x_{n+1} - x_n|$$

for all n. Show that $(x_n)_n$ *is a Cauchy sequence*. (25 pts.)

Proof: Since,

$$|x_{n+2} - x_{n+1}| \leq r\,|x_{n+1} - x_n| \leq r^2\,|x_n - x_{n-1}| \leq r^3\,|x_{n-1} - x_{n-2}| \leq ...$$

one can easily prove by induction that

$$|x_{n+2} - x_{n+1}| \leq r^{n+1}\,|x_1 - x_0|,$$

or that

$$|x_n - x_{n-1}| \leq r^{n-1}\,|x_1 - x_0|.$$

Therefore if $x_1 = x_0$ then the sequence must be a constant sequence, which is clearly Cauchy. Assume from now on that $x_1 \neq x_0$.

Since $\lim_{n\to\infty} r^n = 0$ by question 3, there is an N such that $r^N < \varepsilon(1 - r)/|x_1 - x_0|$. Now for all $n > m > N$ we have,

$$|x_n - x_m| \leq |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + ... + (x_{m+1} - x_m)|$$
$$\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + ... + |x_{m+1} - x_m|.$$
$$\leq r^{n-1}|x_1 - x_0| + r^{n-2}|x_1 - x_0| + ... + r^m|x_1 - x_0|$$
$$= |x_1 - x_0|(r^m + r^{m+1} + ... + r^{n-1}) = |x_1 - x_0|\,r^m(1 + r + ... + r^{n-m-1})$$
$$\leq |x_1 - x_0|\,r^m(1 - r^{n-m})/(1 - r) = |x_1 - x_0|\,r^m/(1 - r)$$
$$\leq |x_1 - x_0|\,r^N/(1 - r) < \varepsilon.$$

Thus the sequence $(x_n)_n$ is Cauchy.