Math 152 Final
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1. Let $f(x) = x^3 + 3/x^2 - 4x + \sqrt{x}$. Show that there is an $a \in [1, 2]$ such that for any $x \in [1, 2]$, $f(x) \geq f(a)$. (4 pts.)

2. Let $f(x) = x^3 - 3/x^2 + 4x + \sqrt{x}$. Show that there is an $a \in (0, 1]$ such that $f(a) = 2$. (4 pts.)

3. Let A and B be two subsets of $\mathbb{R}$ which are bounded above. Let

$$A + B = \{a + b : a \in A, b \in B\}.$$

Show that $\sup(A + B) = \sup A + \sup B$. (7 pts.)

4. Let (X, d) be a metric space. Show that $|d(x, z)| \geq |d(x, y) - d(y, z)|$ for all $x, y, z \in X$. (6 pts.)

5. Let $A \subseteq \mathbb{R}$ be a closed subset of $\mathbb{R}$ which is bounded above. Is it true that $\sup A \in A$? Prove or disprove. (5 pts.)

6. Let (X, d) be a metric space and $A \subseteq X$. Show that A is closed if and only if any convergent sequence $(a_n)_n$ of A converges to an element of A. (10 pts.)

7. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let $d : X_1 \times X_2 \to \mathbb{R}$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

   a) Show that d is a metric on $X_1 \times X_2$. (4 pts.)
   b) Show that a sequence $((x_n, y_n))_n$ coverges to a point (a, b) of $X_1 \times X_2$ if and only if the sequences $(x_n)_n$ and $(y_n)_n$ of X_1 and X_2 converge to the points a and b respectively. (6 pts.)
   c) Show that the topology induced by the metric d on $X_1 \times X_2$ is the product topology. (8 pts.)

8. Let X be a compact space, Y a topological space and $f : X \to Y$ a continuous bijection. Show that f^{-1} is continuous. (5 pts.)

9. Let (X, d) and (Y, d) be two metric spaces and let $f : X \to Y$ be a function. f is said to be locally constant if for any $x \in X$ there is an $\varepsilon > 0$ such that f is constant on the ball $B(x, \varepsilon)$.
   a) Find an example of a locally constant function which is not a constant. (5 pts.)
   b) Show that a locally constant function is continuous. (6 pts.)
   c) Suppose f is locally constant function and $c \in Y$. Show that $\{x \in X : f(x) = c\}$ is both open and closed. (6 pts.)
   d) Show that if X is connected and f is locally constant then f is a constant. (6 pts.)

10. Let X be a topological space. Suppose that for any continuous function $f : X \to \mathbb{R}$ and any real numbers $c < d < e$ if there are $x, y \in X$ such that $f(x) = c$ and $f(y) = d$, then there is a $z \in X$ such that $f(z) = d$. Show that X is connected. (8 pts.)