1. Let \( f(x) = x^3 + 3x^2 - 4x + \sqrt{x} \). Show that there is an \( a \in [1, 2] \) such that for any \( x \in [1, 2] \), \( f(x) \geq f(a) \). (4 pts.)

2. Let \( f(x) = x^3 - 3x^2 + 4x + \sqrt{x} \). Show that there is an \( a \in (0, 1] \) such that \( f(a) = 2 \). (4 pts.)

3. Let \( A \) and \( B \) be two subsets of \( \mathbb{R} \) which are bounded above. Let \( A + B = \{a + b : a \in A, b \in B\} \).
Show that \( \sup(A + B) = \sup A + \sup B \). (7 pts.)

4. Let \((X, d)\) be a metric space. Show that \( |d(x, z)| \geq |d(x, y) - d(y, z)| \) for all \( x, y, z \in X \). (6 pts.)

5. Let \( A \subseteq \mathbb{R} \) be a closed subset of \( \mathbb{R} \) which is bounded above. Is it true that \( \sup A \in A \)? Prove or disprove. (5 pts.)

6. Let \((X, d)\) be a metric space and \( A \subseteq X \). Show that \( A \) is closed if and only if any convergent sequence \((a_n)\) of \( A \) converges to an element of \( A \). (10 pts.)

7. Let \((X_1, d_1)\) and \((X_2, d_2)\) be two metric spaces. Let \( d : X_1 \times X_2 \to \mathbb{R} \) be defined by \( d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \).
   a) Show that \( d \) is a metric on \( X_1 \times X_2 \). (4 pts.)
   b) Show that a sequence \((x_n, y_n)\) converges to a point \((a, b)\) of \( X_1 \times X_2 \) if and only if the sequences \((x_n)\) and \((y_n)\) of \( X_1 \) and \( X_2 \) converge to the points \( a \) and \( b \) respectively. (6 pts.)
   c) Show that the topology induced by the metric \( d \) on \( X_1 \times X_2 \) is the product topology. (8 pts.)

8. Let \( X \) be a compact space, \( Y \) a topological space and \( f : X \to Y \) a continuous bijection. Show that \( f^{-1} \) is continuous. (5 pts.)

9. Let \((X, d)\) and \((Y, d)\) be two metric spaces and let \( f : X \to Y \) be a function. \( f \) is said to be locally constant if for any \( x \in X \) there is an \( \varepsilon > 0 \) such that \( f \) is constant on the ball \( B(x, \varepsilon) \).
   a) Find an example of a locally constant function which is not a constant. (5 pts.)
   b) Show that a locally constant function is continuous. (6 pts.)
   c) Suppose \( f \) is locally constant function and \( c \in Y \). Show that \( \{x \in X : f(x) = c\} \) is both open and closed. (6 pts.)
   d) Show that if \( X \) is connected and \( f \) is locally constant then \( f \) is a constant. (6 pts.)

10. Let \( X \) be a topological space. Suppose that for any continuous function \( f : X \to \mathbb{R} \) and any real numbers \( c < d < e \) if there are \( x, y \in X \) such that \( f(x) = c \) and \( f(y) = d \), then there is a \( z \in X \) such that \( f(z) = d \). Show that \( X \) is connected. (8 pts.)