Math 112 Midterm
Ali Nesin
9 Nisan 2006

1. Let $X = (0, 1)$. Let $C = \{(1/n, 1) : n = 1, 2, ...\}$. Show that any infinite subset of C covers X but that no finite subset of C covers X.

Proof: Let C' be a finite subset of C, $n = \max\{n : (1/n, 1) \in C'\}$ and $x \in (0, 1/n)$. Then $x \notin \cup C'$.

2. Let $T = [0, 1)$ and $0 < \varepsilon < 1$. Let $C = \{(1/n, 1) : n = 1, 2, ...\} \cup \{(-\varepsilon, \varepsilon)\}$. Does C have a finite subcover of T?

3. Let $S \subseteq \mathbb{R}$ and C any cover of S. Show that C has a countable subcover of S.

4. Let $(x_n)_n$ be a converging sequence of a metric space. Let $x = \lim_{n\to\infty} x_n$. Show that

$$\{x_n : n \in \mathbb{N}\} \cup \{x\}$$

is compact.

5. Let X be a metric space. Show that the following are equivalent:

a) X is compact.

b) For any set C of closed subsets of X if $\cap C = \emptyset$ then there are $F(1), ..., F(n) \in C$ such that $F(1) \cap ... \cap F(n) = \emptyset$.

Proof: (a $\Rightarrow$ b). For $F \in C$, let $U_F = F^c$. Then $\cup_{F\in C} U_F = \cup_{F\in C} F^c = (\cap_{F\in C} F)^c = \emptyset^c = X$. Thus $(U_F)_{F\in C}$ is an open cover of X. Since X is compacts, there are $F(1), ..., F(n)$ such that $U_{F(1)} \cup ... \cup U_{F(n)} = X$. By taking the complements, we see that $F(1) \cap ... \cap F(n) = \emptyset$.

(b $\Rightarrow$ a). Let $U = (U_i)_i$ be an open cover of X. Then $\cap_i U_i^c = (\cup_i U_i)^c = X^c = \emptyset$. Thus there $i(1), ..., i(n) \in I$ such that $U_{i(1)}{}^c \cap ... U_{i(n)}{}^c = \emptyset$. By taking the compelements, we see that

5. Let $X \subseteq \mathbb{R}$. Show that the following are equivalent:

a) X is compact.

b) For any set C of closed subsets of X if $\cap C = \emptyset$ then there are $F(1), ..., F(n) \in C$ such that $F(1) \cap ... \cap F(n) = \emptyset$.

c) If $(F(n))_{n\in\mathbb{N}}$ is a sequence of nonempty closed subsets of X such that $F(n+1) \subseteq F(n)$ for all $n \in \mathbb{N}$ then $\cap_n F(n) \neq \emptyset$.

(b $\Rightarrow$ c). Assume $\cap_n F(n) = \emptyset$. Then by hypothesis there are $m_1, ..., m_k$ such that $F(m_1) \cap .. \cap F(m_k) = \emptyset$. Let $n = \max\{m_1, ..., m_k\}$. Then $F(n) = F(1) \cap .. \cap F(n) = \emptyset$, a contradiction.

(c $\Rightarrow$ a). Let $U = (U_i)_i$ be an open cover of X. Assume it has no finite subcover. Let $U_1 \in U$ be one of the open subsets of the cover. Suppose $U_n \in U$ is chosen so that $U_n \not\subset U_1 \cup ... \cup U_{n-1}$. Since $U_1 \cup ... \cup U_{n-1} \cup U_n \neq X$, there is a $U_{n+1} \in U$ so that $U_{n+1} \not\subset U_1 \cup ... \cup U_{n-1} \cup U_n$. This gives us, for each $n \in \mathbb{N}$, $U_n \in U$ such that $U_n \not\subset U_1 \cup ... \cup U_{n-1}$. Let $F_n = (U_1 \cup ... \cup U_n)^c$. Then each F_n is closed, nonempty and $F_{n+1} \subseteq F_n$. Then $\cap_n F_n \neq \emptyset$. By taking

6. Recall that a subset X of a topological space is **connected** if for any two disjoint open subsets U and V of X, if $X \subseteq U \cup V$ then either $X \subseteq U$ or $X \subseteq V$. Show that a connected subset of $\mathbb{R}$ is an interval (of any sort).

7. Show that if $(C_i)_i$ is a family of connected subsets such that $C_i \cap C_j \neq \emptyset$ for all i and j, then $\cup_i C_i$ is also connected.

Proof: Assume U and V are two open disjoint subsets uch that $\cup_i C_i \subseteq U \cup V$. Then for each i, either $C_i \subseteq U$ or $C_i \subseteq V$. If $U_i \subseteq U$ and $U_j \subseteq V$ for i and j then $U_i \cap U_j = \emptyset$, a contradiction. Therefore either all the U_i's are in U or they are all in V. Hence either $\cup_i C_i \subseteq U$ or $\cup_i C_i\ V$.

8. Let $S \subseteq \mathbb{R}$ be a disconnected subset such that $S \cup \{1\}$ is connected. Show that 1 is a limit point of S. (Recall that an element a is a **limit point** of a subset S of a topological space if any open subset containing a contains an element of S different from a.)

Proof: It is enough to show that for any $\varepsilon > 0$, $(1-\varepsilon, 1+\varepsilon) \cap S \neq \emptyset$. Assume this is not the case. Let $\varepsilon > 0$ be such a number. Then $(-\infty, 1-\varepsilon)$ and $(1-\varepsilon/2, \infty)$ disconects $S \cup \{1\}$.

9. Let X be a topological space. Let S be a subset of X. An element $a \in S$ is called **isolated** if there is an open subset U such that $U \cap S = \{a\}$. We let $I(S)$ denote the set of isolated points of S. Find an example of a topological space X and a subset $S \subseteq X$ with $I(S \setminus I(S)) \neq \emptyset$.

10. Show that the product of two compact topological spaces is compact.