Math 151 Final Exam
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1. A metric space (X, d) is called **discrete** if there is an $\alpha > 0$ such that for all distinct $x, y \in X$, $d(x, y) > \alpha$.

1a. Show that any finite metric space is discrete.
1b. Give an example of an infinite discrete metric space.
1c. Show that any Cauchy sequence in a discrete metric space is eventually constant.
1d. Show that any Cauchy sequence in a discrete metric space has a limit.

2. Consider $\mathbb{R}$ with its natural metric space structure.

a) Show that if $s > -1$ then for all natural numbers n, $(1 + s)^n \geq 1 + ns$.
b) Show that if $r \in (-1, 1)$ then the sequence $(r^n)_n$ converges to 0.
c) Show that if $r \notin (-1, 1]$ then the sequence $(r^n)_n$ diverges.
d) Find $\lim_{n \to \infty} (1/n)^n$.
e) Find $\lim_{n \to \infty} (1/2 + 1/n)^n$.

3. Find $\lim_{n \to \infty} \left(\sqrt{n^2 - n} - n\right)$.

4. Find $\lim_{n\to\infty} \left(\dfrac{n-2}{n^2-5}\right)^{\frac{n-1}{3n+5}}$.

5. Show that if $a, b > 0$ then $\lim_{n\to\infty} (a^n + b^n)^{1/n} = \max\{a, b\}$.

6. Let $(x_n)_n$ be a convergent sequence of real numbers. Let

$$y_n = \frac{x_1 + ... + x_n}{n}.$$

Show that $\lim_{n\to\infty} y_n$ exists and is equal to $\lim_{n\to\infty} x_n$.

7. Let $x_1 = 1$, $x_2 = 2$ and $x_n = (x_{n-1} + x_{n-2})/2$ for $n > 2$.

a. Show that $1 \leq x_n \leq 2$ for all n.
b. Show that $|x_n - x_{n+1}| = 1/2^{n-1}$ for all n.
c. Show that if $m > n$ then $x_n - x_m < 1/2^{n-2}$ for all n.
d. Show that $(x_n)_n$ is a Cauchy sequence.
e. Find its $\lim_{n\to\infty} x_n$.

8. We say that a sequence $(x_n)_n$ is **contractive** if there is a constant c, $0 < c < 1$ such that $|x_{n+2} - x_{n+1} \leq c|x_{n+1} - x_n|$ for all n. Show that every contractive sequence is convergent.

9. Let $x_n = 1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2$.

a. Show that for all integers $n \geq 1$, $x_n \leq 2 - 1/n$.
b. Conclude that the sequence $(x_n)_n$ converges.
c. Show that for an integer n large enough, $n^2 \leq 2^n$.
d. Conclude that the sequence $(x_n)_n$ is bounded above by $1 + 1/4 + 1/9 + 1/8 = 107/72$.

10. For a natural number n define $v(n) = \max\{m : 2^m < n\}$. Show that $\lim_{n\to\infty} v(n)/n = 0$.