Local Fields

Summer Midterm I
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Ali Nesin

Throughout, $| \ |$ is a nonarchimedean valuation on a field k.

Let $\mathrm{o}(k) = \mathrm{o} = \{x \in k : |x| \leq 1\}$.

1. Show that o is a subring of k (called the **ring of integers** of the valuation). (2 pts.)

2. Show that the set o^* of invertible elements of o is $\{x \in k : |x| = 1\}$. (2 pts.)

3. Show that $\wp(k) = \wp = \{x \in k : |x| < 1\}$ is an ideal of o. (2 pts.)

4. Show that the ring $\mathrm{o}/\wp$ is in fact a field (called the **residue field** of the valuation). (15 pts.)

5. Let $\underline{k}$ be the completion of k. Denote by $\underline{\mathrm{o}}$ and $\underline{\wp}$ the ring of integers and the corresponding ideal of the field $\underline{k}$. Show that $\mathrm{o} = \underline{\mathrm{o}} \cap k$ and $\wp = \underline{\wp} \cap k$. Deduce that there is a natural one-to-one field homomorphism from $\mathrm{o}/\wp$ into $\underline{\mathrm{o}}/\underline{\wp}$. (2 + 2 + 5 pts.)

6. Show that the above natural map is an isomorphism of fields. (**Hint:** k is dense in $\underline{k}$). (10 pts.)

7. The set $G(k) = G = \{|x| : x \in k^*\}$ is called the **valuation group**. It is clearly a subgroup of $\mathbb{R}^*$. Show that $G(k) = G(\underline{k})$. (10 pts.)

8. We say that the valuation is **discrete** if the valuation group is discrete in the real topology, i.e. if there exists a $\delta > 0$ such that for all $a \in k^*$, if $1 - \delta < |a| < 1 + \delta$ then $|a| = 1$. Show that the valuation is discrete if and only if the ideal $\wp(k)$ is a principal ideal. (15 pts.)

9. Let $k = \mathbb{Q}$ together with the p-adic topology for some prime integer p. Find o, $\wp$ and $\mathrm{o}/\wp$ explicitely. (15 pts.)

10. Let $k = \mathbb{Q}(T)$ together with the T-adic topology. Find o, $\wp$, $\underline{\mathrm{o}}$, $\underline{\wp}$, $\mathrm{o}/\wp$ and $\underline{\mathrm{o}}/\underline{\wp}$ explicitely. (20 pts.)