Math 151 Final
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1. Show that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is continuous. (5 pts.)
2. Let $K$ be a compact metric space. Show that if $f : K \to \mathbb{R}$ is continuous and one to one, then its inverse $f^{-1} : f(K) \to K$ is continuous as well. (10 pts.)
3. A metric space $M$ is said to be **connected** if whenever $M$ is a disjoint union of two open sets, one of the open sets must be empty. Let $M$ and $N$ be two metric spaces and let $f : M \to N$ be a continuous map. Assume that $M$ is connected. Show that $f(M)$ is connected. (10 pts.)
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Let $f^s(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$.

   Show that if $f'(a)$ exists then $f^s(a)$ exists as well. Is the converse true? (5 pts.)
5. Show that if the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0$ and if $g(x_0) \neq 0$ then the function $1/g$ is differentiable at $x_0$. (15 pts.)
6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a \in \mathbb{R}$. Show that $\lim_{x\to a} f(x) = c$ iff for any sequence $(x_n)_n$ that converges to $a$, $\lim_{n\to\infty} f(x_n) = c$. (10 pts.)
7. Show that $\lim_{x\to 0} \sin(1/x)$ does not exist. (5 pts.)
8. Let $(q_n)_n$ be an enumeration of all the rattional numbers. Let $f(q_n) = 1/n$ and $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that $f$ is continuous at all $a \in \mathbb{R} \setminus \mathbb{Q}$. (10 pts.)
9. Let $f : [a, b] \to \mathbb{R}$ be continuous and one to one. Let $c \in (a, b)$ be such that $f$ is differentiable at $c$ and that $f'(c) \neq 0$. Show that $f^{-1} : f([a, b]) \to [a, b]$ is differentiable at $c$ and that $(f^{-1})'(c) = 1/f'(c)$. (**Hint:** You may use #2 and #6). (20 pts.)
10. Integrate $\int_0^{\pi/2} x^2 \sin x dx$, $\int \ln(\sin^2 x)\cos x\, dx$, $\int \frac{xdx}{1+x^2}$, $\int \frac{xdx}{(1+x)^2}$. (2 + 2 + 2 + 3 pts.)
11. Differentiate $\int_x^{\pi/2} \sin t\, dt$ $\int_0^{x^3} \sin t\, dt$ and $\int_{x^2}^{x^3} \sin t\, dt$ with respect to $x$. (2 + 3 + 4 pts.)