Analysis Final Exam (Math 212)
Spring 2005
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1. Find a sequences $(a_n)_n$ of real numbers such that any real number is the limit of one of its subsequences.

Recall that given a sequence $(a_n)_n$ of real numbers limsup a_n is defined as

$$\text{limsup } a_n = \lim_{n\to\infty} \sup\{a_m : m > n\}.$$

(Here $\sup\{a_m : m > n\}$ can be ∞ and in this case $\lim_{n\to\infty} \infty$ is defined to be ∞). Note that limsup a_n can be a real number as well as one of ∞ and $-\infty$.

2a. Show that limsup $a_n = \infty$ if and only if $(a_n)_n$ is unbounded.

2b. Show that limsup $a_n = -\infty$ if and only if $\lim a_n = -\infty$.

Proof: Since the sequence $(\sup\{a_m : m > n\})_n$ is nonincreasing limsup $a_n = \infty$ if and only if $\sup\{a_m : m > n\} = \infty$ for all n. This means that given any $r \in \mathbb{R}$, there is an m such that $r < a_m$. Hence the sequence $(a_n)_n$ is unbounded. The converse is easy.

Extending the previous definition of convergence, if $\lim_{n\to\infty} a_n = \infty$, we will say that the sequences $(a_n)_n$ converges to ∞. Similarly if $\lim_{n\to\infty} a_n = -\infty$, we will say that the sequences $(a_n)_n$ converges to $-\infty$.

3. Show that limsup a_n is the supremum of the set of limits of convergent subsequences of $(a_n)_n$.

Proof: Note that limsup $a_n = \infty$ if and only if the sequence $(a_n)_n$ is unbounded. Also, the sequence $(a_n)_n$ is unbounded if and only if $(a_n)_n$ has a subsequence converging to ∞. This proves the result in case limsup $a_n = \infty$.

Assume now limsup $a_n = -\infty$. Then for any $r \in \mathbb{R}$ there is an integer N such that for all $n > N$, $a_n < r$. It follows that no subsequence of $(a_n)_n$ can converge to $r + 1$ or to ∞. Therefore no subsequence of $(a_n)_n$ can converge to a real number or to ∞. Also, given a $k \in \mathbb{N}$, choose an integer n_k such that $n_k > n_{k-1}$ and $a_{n_k} < -k$. Then the subsequence $(a_{n_k})_k$ converges to $-\infty$.

Conversely suppose that no subsequence of $(a_n)_n$ can converge to a real number or to ∞. Then the sequence $(a_n)_n$ is bounded above (otherwise one of its subsequences will converge to ∞), say by M. It follows that given an $r \in \mathbb{R}$, $\{n : a_n \in [r, \infty)\}$ is finite (otherwise for infinitely many n, a_n will be in the closed interval $[r, M]$ and there will be a convergent subsequence), thus there is an N such that for all $n > N$, $a_n < r$. Thus the sequence

Let $(b_n)_n$ subsequence of $(a_n)_n$ converging to $r \in \mathbb{R} \cup \{\pm\infty\}$.

If $r = \infty$, then the terms of the subsequence $(b_n)_n$ are unbounded, therefore the terms of the sequence $(a_n)_n$ are unbounded as well. Therefore

7. Let $\Sigma_n a_nx^n$ be a power series. Let $R = 1/\text{limsup } |a_n|^{1/n}$. Show that $\Sigma_n a_nx^n$ converges absolutely for $|x| < R$ and diverges for $|x| > R$. (Recall that if $b_n \geq 0$ for all n then limsup b_n is defined as the limit of the nonincreasing sequence $\sup\{b_i : i > n\} \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ and $1/\infty$ and $1/0$ are defined to be 0 and ∞ respectively. R is called the radius of convergence of the series $\Sigma_n a_nx^n$).

Proof: Let $r = \limsup |a_n|^{1/n} < 1$. Let $\varepsilon = (1 - r)/2$ and $\rho = r + \varepsilon$. Then $\rho = r + \varepsilon = r + (1-r)/2 = (1 + r)/2 < 1$. Since $\varepsilon > 0$, the nonincreasing sequence $\sup\{|a_n|^{1/n} : i > n\} < r + \varepsilon$ after a while. i.e. $|a_n|^{1/n} \in [r, r + \varepsilon)$

8. Let $0 \leq S < R$. Show that $\Sigma_n a_n x^n$ converges uniformly on $\{x : |x| \leq S\}$.

9. Show that the radius of convergence of the power series $\Sigma_{n\geq 0} a_n x^n$ and $\Sigma_{n>0} na_n x^{n-1}$ are the same.

10. Let $\Sigma_n a_n x^n$ be a power series and R its radius of convergence. Show that the series is differentiable at any x such that $|x| < R$.