Math 152 Midterm

Ali Nesin
March 2005

1. By using the definition of continuity, show that the function $f(x) = \dfrac{x+4}{x^2-1}$ is continuous in its domain of definition.

2. Given two continuous real valued functions f and g from a metric space X show that $\max\{f(x), g(x)\}$ is also continuous.

A sequence of functions $(f_n)_n$ from a set X into a metric space Y is said to *converge uniformly* to a function $f : X \to Y$ if for any $\varepsilon > 0$ there is an N such that for all $n > N$ and all $x \in X$, $d(f_n(x), f(x)) < \varepsilon$. A sequence of functions $(f_n)_n$ from a set X into a metric space Y is said to *converge pointwise* to a function $f : X \to Y$ if for any $x \in X$ and $\varepsilon > 0$ there is an N such that for all $n > N$, $d(f_n(x), f(x)) < \varepsilon$.

3a. Show that uniform convergence implies pointwise convergence.

3b. Let $X = [0, 1]$, $Y = \mathbb{R}$ and $f_n(x) = x^n$. Show that $(f_n)_n$ converges pointwise but not uniformly.

3c. Let $X = [0, a]$ where $0 \leq a < 1$, $Y = \mathbb{R}$ and $f_n(x) = x^n$. Show that $(f_n)_n$ converges uniformly.

4. Let $X = \mathbb{R} \setminus \{-1\}$ $Y = \mathbb{R}$ and $f_n(x) = \dfrac{1}{1+x^n}$.

4a. Find the set $A = \{x \in \mathbf{R} : (f_n(x))_n \text{ converges}\}$.

For $x \in A$, let $f(x) = \lim_{n \to \infty} f_n(x)$.

4b. What is f?

4c. Discuss the uniform convergence of $(f_n)_n$ in the (open or closed) intervals contained in A.

5. Let $X = [0, 1]$, $Y = \mathbb{R}$ and $f_n(x) = \dfrac{nx}{x+n}$ for $n > 1$. Show that the sequence $(f_n)_n$ converges uniformly to some function.

6. Let X be a set and Y a metric space. Let B be the set of bounded functions from X into Y, i.e., $B = \{f : X \to Y : f(X) \text{ is bounded}\}$. Find a metric on B such that the uniform convergence corresponds exactly to the convergence in this metric.

7\. Let X be a set and Y a metric space. Let $(f_n)_n$ converge uniformly to some f. Assume that $a \in X$ is such that each f_n is continuous at a. Show that f is continuous at a.