Math 151  
Fall 2004 Resit Exam on Real Powers  
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No use of logical symbols such as $\forall$, $\exists$, $\Rightarrow$ is allowed, 1 point out of 100 will be taken for each use of these symbols.

Explain your work. Make complete and correct sentences with at least one subject and one verb.

**Part 0.**

1. Show that for $x, a \in \mathbb{R}^{>0}$ and $n \in \mathbb{N}^{>0}$, $x > a$ iff $x^n > a^n$. (Here $x^n$ stands for $x$ multiplied with itself $n$ times, i.e. for $x$ if $n = 1$ and for $x \cdot x^{n-1}$ if $n > 1$). (5 pts.)

**Proof:** Assume $x > a$. We show $x^n > a^n$. If $n = 1$ this is clear. For $n > 1$, by induction on $n$ we get $x^n = x \cdot x^{n-1} > x \cdot a^{n-1} > a \cdot a^{n-1} = a^n$. Conversely assume $x^n > a^n$. If $x \leq a$ then by the first part $x^n \leq a^n$, a contradiction.

2. Show that if $a \geq b$ are real numbers with $a \geq 0$ and $n$ is a natural number then $(a - b)^n \geq a^n - nd^{n-1}b$.

(5 pts.)

**Proof:** By induction on $n$. If $n = 0$ that is clear. If $n > 0$:

$$(a - b)^n = (a - b)^{n-1}(a - b) \geq (a^{n-1} - (n-1)a^{n-2}b)(a - b) = a^n - (n-1)a^{n-1}b - a^{n-1}b + (n-1)a^{n-2}b^2 \geq a^n - (n-1)a^{n-1}b - a^{n-1}b = a^n - nd^{n-1}b.$$

3. Let $a, x > 0$ be real numbers and $n$ a positive natural number. Suppose $x^n < a$. Show that for some $\delta > 0$, $(x + \delta)^n \leq a$. (10 pts.)

**Proof:** Suppose $x \geq 0$ is such that $x^n < a$. Let $M = \max\{x, x^2, ..., x^n\} + 1$. Let

$$\tilde{\delta} = \max\left\{\frac{1}{2^n}, \frac{a - x^n}{Mn!(n-1)}\right\}.$$

Then

$$(x + \tilde{\delta})^n = \sum_{i=0}^{n} \binom{n}{i} x^i \tilde{\delta}^{n-i} = x^n + \sum_{i=1}^{n} \binom{n}{i} x^i \tilde{\delta}^{n-i} \leq x^n + \sum_{i=1}^{n} \binom{n}{i} x^i \tilde{\delta} \leq x^n + \sum_{i=1}^{n} \binom{n}{i} M \tilde{\delta} \leq x^n + \sum_{i=1}^{n} \frac{a - x^n}{n-1} = x^n + (a - x^n) = a.$$

4. Let $a, x > 0$ be real numbers and $n$ a positive natural number. Suppose $a < x^n$. Show that there is a $\delta > 0$ such that $a < (x - \delta)^n$. (10 pts.)

**Proof:** Suppose now $x \geq 0$ is such that $a < x^n$. Let $\delta = \max\left\{\frac{x}{2}, \frac{x^n - a}{2nx^n}\right\}$. Then, by Q2,

$$(x - \delta)^n \geq x^n - n\delta x^{n-1} = x^n - (x^n - a)/2 > x^n - (x^n - a) = a.$$
Part I. Let $a \geq 1$ be a real number.

5. For a positive integer $n$, show that the set $A(a, n) := \{x \in \mathbb{R} : x^n \leq a\}$ is bounded above. (3 pts.)

We let $a^{(1,n)} := \sup A(a, n)$. Note that $a^{(1,n)}$ is supposed to mean $a^{1/n}$ (See Q7). We will soon change our notation to this standard notation (see Q11).

Proof. Suppose $x \in A(a, n)$. Then $x^n \leq a$ because $a > 1$. Therefore $x^n \leq a^n$. It follows from Q1 that $x \leq a$. Hence $A(a, n)$ is bounded above by $a$.

6. Show that $a^{(1,1)} = a$. (2 pts.)

Proof: By definition $a^{(1,1)} := \sup A(a, 1) = \sup \{x \in \mathbb{R} : x \leq a\} = a$.

7. Show that $(a^{(1,n)})^n = a$. (5 pts.)

Proof: By definition $(a^{(1,n)})^n := \sup A(a^n, n) = \sup \{x \in \mathbb{R} : x^n \leq a^n\} = \sup \{x \in \mathbb{R} : x \leq a\} = a$ by Q1. Hence $(a^{(1,n)})^n = a$.

8. Show that $(a^{(1,n)})^n = a$. (10 pts.)

Proof: If $(a^{(1,n)})^n < a$, then by taking $x = a^{(1,n)}$ in Q3 we see that $a^{(1,n)} + \delta \in A(a, n)$ for some $\delta > 0$. But this contradicts the definition of $a^{(1,n)}$. Hence $(a^{(1,n)})^n \geq a$.

If $(a^{(1,n)})^n > a$, then by taking $x = a^{(1,n)}$ in Q4 we see that $(a^{(1,n)} - \delta)^n > a$ for some $a^{(1,n)} > \delta > 0$. But since $a^{(1,n)} - \delta < a^{(1,n)} = \sup A(a, n)$, there is a $b \in A(a, n)$ such that $a^{(1,n)} - \delta \leq b$ and $a < (a^{(1,n)} - \delta)^n \leq b^n \leq a$, a contradiction.

9. Show that $(a^{(1,n)})^m = (a^{m(1,n)})$. (10 pts.)

Proof: $((a^{(1,n)})^m)^n = ((a^{(1,n)})^m)^m = a^m$ by Q8. Also $(a^{(1,n)})^m = a^m$ by Q7. Thus $(a^{(1,n)})^m = (a^{m(1,n)})$. By Q1, we get $(a^{(1,n)})^m = (a^{m(1,n)})$.

10. Show that if $n|m = plq$ then $(a^{(p,q)}) = (a^{(1,q)})^{mp} = (a^{(1,q)})^{np}$. (10 pts.)

Proof: $(a^{(p,q)})^m = (a^{(1,q)})^{mp} = (a^{(1,q)})^{np}$ by Q7 and Q8. Thus $(a^{(1,m)}) = (a^{(1,q)})^{np}$ by Q1.

11. Deduce that for any positive rational number $n/m$ (with $n, m > 0$) we are allowed to define $a^{n/m}$ as the real number $a^{(1,m)}$. Show that if $n|m = k \in \mathbb{N}$ then $a^{n/m} = a^k$. (Here $a^k$ means a multiplied with itself $k$ times). (5 pts.)

Proof: The first part is from Q10. Since $n|m = k/1$, it is enough to show that $a^{k/1} = a^k$. But, $a^{k/1} = (a^{k(1,1)})^{1/1} = a^k$ by Q6.

12. Show that for positive rational numbers $p$ and $q$, $a^p/q = (a^p)^q$. (10 pts.)

Proof: Writing $p = n/m$ and $q = r/s$ with $n, m, p, q$ positive natural numbers, we see that we have to show $a^{n/m} = (a^{n/m})^{r/s}$. By definition this means $a^{n/m} = ((a^{n/m})^{r/s})^{r/s}$. By Q9 this means $a^{n/m} = ((a^{n/m})^{r/s})^{r/s}$.

By Q1 this means $a^{n/m} = (a^{(1,m)})^{r/s}$. By Q1 again this means $a^{n/m} = (a^{(1,m)})^{r/s}$. Finally by Q8 and Q9 this means $a = a$, which certainly holds.

13. Show that any real numbers $a, b \geq 1$ and a positive rational number $p$, $(a)^p = a^p b^p$. (5 pts.)

Proof: Writing $p = n/m$ with $n, m \in \mathbb{N} > 0$, we need to show that $((a)^n)^{1/m} = (a^p)^{1/m} = (b^p)^{1/m}$. By Q1 and Q8, we need to show that $(a)^{p} = a^p b^p$, which certainly holds.
14. Show that for positive rational numbers $p$ and $q$, $a^{p+q} = a^p a^q$. (5 pts.)

Proof: Writing $p = \frac{n}{m}$ and $q = \frac{r}{s}$ with $n, m, p, q$ positive natural numbers, we see that we have to show that $a^{\left(\frac{ns}{ms}+\frac{mr}{ms}\right)} = a^{\frac{nm}{ms}}a^{\frac{rs}{ms}}$. By Q12 and Q13, taking the $ms$th power, this means $a^{\frac{ns+mr}{ms}} = a^{ns}a^{mr}$, which certainly holds.

Part 2. Let $a \geq 1$ be a real number.

15. For a positive real number $r$, show that the set $\{a^q : 0 < q \leq r$ and $q \in \mathbb{Q} \}$ is bounded above. We let $a^{(r)} = \sup\{a^q \in \mathbb{R} : q \leq r\}$.

16. Show that $a^{(p)} = a^p$ for any positive rational number $p$. From now on we denote $a^{(r)}$ as $a^r$.

17. Show that for positive real numbers $r$ and $s$, $a^{rs} = a^r a^s$, $a^{rs} = (a^r)^s$ and $(ab)^r = a^r b^r$.

18. Show that any real numbers $a, b \geq 1$ and a positive rational number $p$, $(ab)^p = a^p b^p$. 