Math 151
Fall 2004 Resit Exam on Real Powers
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No use of logical symbols such as $\forall$, $\exists$, $\Rightarrow$ is allowed, 1 point out of 100 will be taken for each use of these symbols.

Explain your work. Make complete and correct sentences with at least one subject and one verb.

Part 0.

1. *Show that for* $x, a \in \mathbb{R}^{>0}$ *and* $n \in \mathbb{N}^{>0}$, $x > a$ *iff* $x^n > a^n$. (*Here* x^n *stands for* x *multiplied with itself* n *times, i.e. for* x *if* $n = 1$ *and for* $x.x^{n-1}$ *if* $n > 1$). (5 pts.)

Proof: Assume $x > a$. We show $x^n > a^n$. If $n = 1$ this is clear. For $n > 1$, by induction on n we get $x^n = x.x^{n-1} > x.a^{n-1} > a.a^{n-1} = a^n$. Conversely assume $x^n > a^n$. If $x \leq a$ then by the first part $x^n \leq a^n$, a contradiction.

2. *Show that if* $a \geq b$ *are real numbers with* $a \geq 0$ *and* n *is a natural number then*

$$(a - b)^n \geq a^n - na^{n-1}b.$$

(5 pts.)

Proof: By induction on n. If $n = 0$ that is clear. If $n > 0$:
$(a - b)^n = (a - b)^{n-1}(a - b) \geq (a^{n-1} - (n-1)a^{n-2}b)(a - b) = a^n - (n-1)a^{n-1}b - a^{n-1}b + (n-1)a^{n-2}b^2 \geq a^n - (n-1)a^{n-1}b - a^{n-1}b = a^n - na^{n-1}b.$

3. *Let* $a, x > 0$ *be real numbers and* n *a positive natural number. Suppose* $x^n < a$. *Show that for some* $\delta > 0$, $(x + \delta)^n \leq a$. (10 pts.)

Proof: Suppose $x \geq 0$ is such that $x^n < a$. Let $M = \max\{x, x^2, ..., x^n\}+1$. Let $\delta = \max\left\{\frac{1}{2}, \frac{a - x^n}{Mn!(n-1)}\right\}$. Then

$$(x+\delta)^n = \sum_{i.=0}^{n}\binom{n}{i}x^i\delta^{n-i} = x^n + \sum_{i.=1}^{n}\binom{n}{i}x^i\delta^{n-i} \leq x^n + \sum_{i.=1}^{n}\binom{n}{i}x^i\delta \leq x^n + \sum_{i.=1}^{n}\binom{n}{i}M\delta$$

$$\leq x^n + \sum_{i.=1}^{n} n!M\delta \leq x^n + \sum_{i.=1}^{n}\frac{a - x^n}{n-1} = x^n + (a - x^n) = a.$$

4. *Let* $a, x > 0$ *be real numbers and* n *a positive natural number. Suppose* $a < x^n$. *Show that there is a* $\delta > 0$ *such that* $a < (x - \delta)^n$. (10 pts.)

Proof: Suppose now $x \geq 0$ is such that $a < x^n$. Let $\delta = \max\left\{\frac{x}{2}, \frac{x^n - a}{2nx^{n-1}}\right\}$. Then, by Q2,

$(x - \delta)^n \geq x^n - n\delta x^{n-1} = x^n - (x^n - a)/2 > x^n - (x^n - a) = a.$

Part I. *Let $a \geq 1$ be a real number.*

5. *For a positive integer n, show that the set $A(a, n) := \{x \in \mathbb{R} : x^n \leq a\}$ is bounded above.* (3 pts.)

We let $a^{(1, n)} := \sup A(a, n)$. Note that $a^{(1, n)}$ is supposed to mean $a^{1/n}$ (See Q7). *We will soon change our notation to this standard notation* (*see* Q11).

Proof. Suppose $x \in A(a, n)$. Then $x^n \leq a \leq a^n$ because $a > 1$. Therefore $x^n \leq a^n$. It follows from Q1 that $x \leq a$. Hence $A(a, n)$ is bounded above by a.

6. *Show that $a^{(1,1)} = a$.* (2 pts.)

Proof: By definition $a^{(1, 1)} := \sup A(a, 1) = \sup\{x \in \mathbb{R} : x \leq a\} = a$.

7. *Show that $(a^n)^{(1, n)} = a$.* (5 pts.)

Proof: By definition $(a^n)^{(1, n)} := \sup A(a^n, n) = \sup\{x \in \mathbb{R} : x^n \leq a^n\} = \sup\{x \in \mathbb{R} : x \leq a\} = a$ by Q1. Hence $(a^n)^{(1, n)} = a$.

8. *Show that $(a^{(1, n)})^n = a$.* (10 pts.)

Proof: If $(a^{(1, n)})^n < a$, then by taking $x = a^{(1, n)}$ in Q3 we see that $a^{(1, n)} + \delta \in A(a, n)$ for some $\delta > 0$. But this contradicts the definition of $a^{(1, n)}$. Hence $(a^{(1, n)})^n \geq a$.

If $(a^{(1, n)})^n > a$, then by taking $x = a^{(1, n)}$ in Q4 we see that $(a^{(1, n)} - \delta)^n > a$ for some $a^{(1, n)} > \delta > 0$. But since $a^{(1, n)} - \delta < a^{(1, n)} = \sup A(a, n)$, there is a $b \in A(a, n)$ such that $a^{(1, n)} - \delta \leq b \leq a^{(1, n)}$. Then $a < (a^{(1, n)} - \delta)^n \leq b^n \leq a$, a contradiction.

9. *Show that $(a^{(1, n)})^m = (a^m)^{(1, n)}$.* (10 pts.)

Proof: $((a^{(1, n)})^m)^n = ((a^{(1, n)})^n)^m = a^m$ by Q8. Also $((a^m)^{(1, n)})^n = a^m$ by Q7. Thus $((a^{(1, n)})^m)^n = ((a^m)^{(1, n)})^n$. By Q1, we get $(a^{(1, n)})^m = (a^m)^{(1, n)}$.

10. *Show that if $n/m = p/q$ then $(a^n)^{(1, m)} = (a^p)^{(1, q)}$.* (10 pts.)

Proof: $((a^p)^{(1, q)})^{mp} = ((a^p)^{(1, q)})^{nq} = a^{np} = ((a^n)^{(1, m)})^{mp}$ by Q7 and Q8. Thus $(a^n)^{(1, m)} = (a^p)^{(1, q)}$ by Q1.

11. *Deduce that for any positive rational number n/m* (*with n, $m > 0$*) *we are allowed to define $a^{n/m}$ as the real number $(a^n)^{(1, m)}$. Show that if $n/m = k \in \mathbb{N}$ then $a^{n/m} = a^k$.* (*Here a^k means a multiplied with itself k times*). (5 pts.)

Proof: The first part is from Q10. Since $n/m = k/1$, it is enough to show that $a^{k/1} = a^k$. But, $a^{k/1} = (a^k)^{(1, 1)} = a^k$ by Q6.

12. *Show that for positive rational numbers p and q, $a^{pq} = (a^p)^q$.* (10 pts.)

Proof: Writing $p = n/m$ and $q = r/s$ with n, m, p, q positive natural numbers, we see that we have to show $a^{nr/ms} = (a^{n/m})^{r/s}$. By definition this means $(a^{rn})^{(1, ms)} = (((a^n)^{(1, m)})^r)^{(1, s)}$. By Q9 this means $(a^{(1, ms)})^{rn} = (((a^{(1, m)})^{(1, s)})^{rn}$. By Q1 this means $a^{(1, ms)} = (a^{(1, m)})^{(1, s)}$. By Q1 again this means $(a^{(1, ms)})^{ms} = ((a^{(1, m)})^{(1, s)})^{ms}$. Finally by Q8 and Q9 this means $a = a$, which certainly holds.

13. *Show that any real numbers a, $b \geq 1$ and a positive rational number p, $(ab)^p = a^p b^p$.* (5 pts.)

Proof: Writing $p = n/m$ with n, $m \in \mathbb{N}^{>0}$, we need to show that $((ab)^n)^{(1,m)} = (a^n)^{(1,m)} (b^n)^{(1,m)}$. By Q1 and Q8, we need to show that $(ab)^n = a^n b^n$, which certainly holds.

14. *Show that for positive rational numbers p and q*, $a^{p+q} = a^p a^q$. (5 pts.)

Proof: Writing $p = n/m$ and $q = r/s$ with n, m, p, q positive natural numbers, we see that we have to show that $a^{(ns+mr)/ms} = a^{n/m}a^{r/s}$. By Q12 and Q13, taking the ms^{th} power, this means $a^{ns+mr} = a^{ns}a^{mr}$, which certainly holds.

Part 2. *Let* $a \geq 1$ *be a real number.*

15. *For a positive real number r, show that the set* $\{a^q : 0 < q \leq r \text{ and } q \in \mathbb{Q}\}$ *is bounded above. We let*

$$a^{(r)} = \sup\{a^q \in \mathbb{R} : q \leq r\}.$$

16. *Show that* $a^{(p)} = a^p$ *for any positive rational number p. From now on we denote* $a^{(r)}$ *as* a^r.

17. *Show that for positive real numbers r and s*, $a^{r+s} = a^r a^s$, $a^{rs} = (a^r)^s$ and $(ab)^r = a^r b^r$.

18. *Show that any real numbers* $a, b \geq 1$ *and a positive rational number p*, $(ab)^p = a^p b^p$.