Math 151 Final (2)

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A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous at a point** $a \in \mathbf{R}$, if for all $\varepsilon > 0$ (real or rational, it does not matter) there is a $\delta > 0$ such that for all $x \in \mathbf{R}$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous** if it is continuous at every point $a \in \mathbb{R}$.

1. *Show that a constant function is continuous*. (3 pts.)

Proof. Let f be a constant function, say $f(x) = c$ for all $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. We will show that f is continuous at a. Let $\varepsilon > 0$. Choose $\delta = 1$ (or any positive real number). Assume $x \in \mathbb{R}$ is such that $|x - a| < \delta$. Then $|f(x) - f(a)| = |c - c| = 0 < \varepsilon$. Hence f is continuous at a. Since this holds for all $a \in \mathbb{R}$, f is continuous.

2. *Show that the identity function is continuous*. (3 pts.)

Proof. Let $a \in \mathbb{R}$. We will show that Id is continuous at a. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$ (or any positive number less than ε). Assume $x \in \mathbb{R}$ is such that $|x - a| < \delta$. Then $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$. Hence f is continuous at a. Since this holds for all $a \in \mathbb{R}$, f is continuous.

3. *Is the function f defined by*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

continuous? *Justify your answer*. (4 pts.)

Answer: This function is continuous everywhere except at $a = 0$. Indeed let $\varepsilon = 1$ (or any positive real number less than 2). Let $\delta > 0$ be any positive real number. Choose $x = -\delta/2$. Then $|x - a| = |x| = |-\delta/2| = \delta/2 < \delta$ but $|f(x) - f(a)| = |f(x) - f(0)| = |f(x) - 1| = |f(-\delta/2) - 1| = |-1 - 1| = 2 > 1 = \varepsilon$.

4. *Let f be defined as follows:*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise} \end{cases}$$

Is f continuous at some point? Justify your answer. (5 pts.)

Proof: No, f is not continuous anywhere. Indeed let $a \in \mathbb{R}$ be any real number. Let $\varepsilon = 1$ (or any positive real number less than 1). Let $\delta > 0$ be any. Choose $x \in (a - \delta, a + \delta)$ such that x is rational if a is irrational and x is irrational if a is rational. (Since is dense in $\mathbb{R}$, there is such an x). Now $|f(x) - f(a)| = 1 = \varepsilon \geq \varepsilon$. Thus f is not continuous at a.

5. *Show that if f and g are continuous, then so is their sum f + g*. (6 pts.)

Proof. Let $a \in \mathbb{R}$. Let $\varepsilon > 0$. Since f is continuous, there is a $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta_1$ then $|f(x) - f(a)| < \varepsilon/2$. Similarly, snce g is continuous, there is a $\delta_2 > 0$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta_1$ then $|g(x) - g(a)| < \varepsilon/2$. Now let $\delta = \min(\delta_1, \delta_2)$. For any $x \in \mathbb{R}$ such that $|x - a| < \delta$, we have, $|(f+g)(x) - (f+g)(a)| = |(f(x) + g(x)) - (f(a) + g(a))| = |(f(x) - f(a)) + (g(x) + g(a))| \le |f(x) - f(a)| + |g(x) + g(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

6. *Show that if f and g are continuous, then so is their product f • g.* (7 pts.)

Proof. Let $a \in \mathbb{R}$. Let $\varepsilon > 0$.

Since f is continuous at a, there is a δ_1 such that if $|x - a| < \delta_1$ then $|f(x) - f(a)| < 1$, i.e. $f(a) - 1 < f(x) < f(a) + 1$. Let $M = \max(|f(a) - 1|, |f(a) + 1|)$. Then for all $x \in \mathbb{R}$ for which $|x - a| < \delta_1$, we have $|f(x)| < M$.

Let N be such that $|g(a)| < N$.

Since f is continuous at a there is a $\delta_2 > 0$ such that for all $x \in \mathbb{R}$ for which $|x - a| < \delta_2$, we have $|f(x) - f(a)| < \varepsilon/2N$.

Since g is continuous at a there is a $\delta_3 > 0$ such that for all $x \in \mathbb{R}$ for which $|x - a| < \delta_3$, we have $|g(x) - g(a)| < \varepsilon/2M$.

Now let $\delta = \max(\delta_1, \delta_2, \delta_3)$. Then for all $x \in \mathbb{R}$ for which $|x - a| < \delta_2$, we have: $|(fg)(x) - (fg)(a)| = |f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \le |f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| = |f(x)||g(x) - g(a)| + |f(x) - f(a)||g(a)| < M(\varepsilon/2M) + (\varepsilon/2N)N = \varepsilon$.

7. *By applying the previous questions show that the function defined by*

$$f(x) = x^2 - 4x + \sqrt{2}$$

is continuous. (5 pts.)

Proof: The identity function is continuous by Q2. The squaring function $g(x) = x^2$ is continuous from this and Q6. By multiplying the constant -4 function with Id, by Q1 and Q6, we see that $h(x) = -4x$ is continuous as well. The constant function $k(x) = \sqrt{2}$ being also continıous, by Q5, $f = g + h + k$ is continuous as well.

8. *By using directly the definition of continuity show that the function defined by* $f(x) = x^2 - 4x + \sqrt{2}$ *is continuous*. (7 pts.)

Proof: Let $a \in \mathbb{R}$, $\varepsilon > 0$, $M = \max(|a - 1|, |a + 1|)$ and $\delta = \min(1, \varepsilon/(M + a + 4))$. Then if $x \in \mathbb{R}$ is such that $|x - a| < \delta$, then $-\delta < x - a < \delta$ and $a - 1 \le a - \delta < x < a + \delta \le a + 1$, and so $|x| < M$. Hence

$$\begin{aligned} |(x^2 - 4x + \sqrt{2}) - (a^2 - 4a + \sqrt{2})| &= |(x^2 - a^2 + (4a - 4x)| = |x - a||x + a - 4| \\ &< |x - a|(|x| + |a| + 4) < |x - a|(M + a + 4) < \varepsilon. \end{aligned}$$

9. *Let f and g be two functions. Assume that f is continuous at a and that g is continuous at f(a). Show that* $g \circ f$ *is continuous at a*. (10 pts.)

Proof. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Since g is continuous at $f(a)$, there is a $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $|x - f(a)| < \delta_1$ then $|g(f(x)) - g(f(a))| < \varepsilon$. Since f is continuous at a there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \delta_1$. Now for $x \in \mathbb{R}$ that satisfies $|x - a| < \delta$, we have first $|f(x) - f(a)| < \delta_1$ and then then $|g(f(x)) - g(f(a))| < \varepsilon$.

10. Show that exp x defined by $\sum_{n\geq 0} x^n/n!$ is continuous. (15 pts.)

Proof. We know that exp x exists (i.e. the series converges for all x). Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Let $M = \max(|a-1|, |a+1|)$ and $\delta = \min(1, \varepsilon/\exp M)$. Now assume $x \in \mathbb{R}$ satisfies $|x - a| < \delta$. We have, first of all $|x| < M$ as in Q8. It is easy to show that $|a| < M$ as well. Finally,

$$\begin{aligned}
|\textstyle\sum_{n\geq 0} x^n/n! - \sum_{n\geq 0} a^n/n!| &= |\textstyle\sum_{n\geq 0} (x^n - a^n)/n!| = |\sum_{n\geq 1} (x^n - a^n)/n!| \\
&= \textstyle\sum_{n\geq 1} |x^n - a^n|/n! = \sum_{n\geq 1} |x^n - a^n|/n! \\
&= |x-a| \textstyle\sum_{n\geq 1} |x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}|/n! \\
&\leq |x-a| \textstyle\sum_{n\geq 1} nM^{n-1}/n! = |x-a| \sum_{n\geq 1} M^{n-1}/(n-1)! \\
&= |x-a| \exp M < \varepsilon
\end{aligned}$$

11. *Let f be continuous at a and assume that $f(a) > 0$. Show that there is an interval around a where f is strictly positive.* (15 pts.)

Proof. Since f is continuous, by choosing $\varepsilon = f(a)/2 > 0$, we find a δ such that for $x \in (a-\delta, a+\delta)$, $f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon) = (f(a)/2, 3f(a)/2) \subseteq \mathbb{R}^{>0}$.

12. *Let f be continuous. Assume that $f(a) < 0$ and $f(b) > 0$. Show that $f(c) = 0$ for some c between a and b.* (15 pts.)

13. Generalize the concept of a continuous function to metric spaces. (10 pts.)