A function \( f : \mathbb{R} \to \mathbb{R} \) is called **continuous at a point** \( a \in \mathbb{R} \), if for all \( \varepsilon > 0 \) (real or rational, it does not matter) there is a \( \delta > 0 \) such that for all \( x \in \mathbb{R} \), if \( \lvert x - a \rvert < \delta \) then \( \lvert f(x) - f(a) \rvert < \varepsilon \).

A function \( f : \mathbb{R} \to \mathbb{R} \) is called **continuous** if it is continuous at every point \( a \in \mathbb{R} \).

1. **Show that a constant function is continuous.** (3 pts.)

   **Proof.** Let \( f \) be a constant function, say \( f(x) = c \) for all \( x \in \mathbb{R} \). Let \( a \in \mathbb{R} \). We will show that \( f \) is continuous at \( a \). Let \( \varepsilon > 0 \). Choose \( \delta = 1 \) (or any positive real number). Assume \( x \in \mathbb{R} \) is such that \( \lvert x - a \rvert < \delta \). Then \( \lvert f(x) - f(a) \rvert = \lvert c - c \rvert = 0 < \varepsilon \). Hence \( f \) is continuous at \( a \). Since this holds for all \( a \in \mathbb{R} \), \( f \) is continuous.

2. **Show that the identity function is continuous.** (3 pts.)

   **Proof.** Let \( a \in \mathbb{R} \). We will show that \( \text{Id} \) is continuous at \( a \). Let \( \varepsilon > 0 \). Choose \( \delta = \varepsilon \) (or any positive number less than \( \varepsilon \)). Assume \( x \in \mathbb{R} \) is such that \( \lvert x - a \rvert < \delta \). Then \( \lvert f(x) - f(a) \rvert = \lvert x - a \rvert < \delta = \varepsilon \). Hence \( f \) is continuous at \( a \). Since this holds for all \( a \in \mathbb{R} \), \( f \) is continuous.

3. **Is the function \( f \) defined by**

   \[
   f(x) = \begin{cases} 
   -1 & \text{if } x < 0 \\
   1 & \text{if } x \geq 0
   \end{cases}
   \]

   **continuous? Justify your answer.** (4 pts.)

   **Answer:** This function is continuous everywhere except at \( a = 0 \). Indeed let \( \varepsilon = 1 \) (or any positive real number less than 2). Let \( \delta > 0 \) be any positive real number. Choose \( x = -\delta/2 \). Then \( \lvert x - a \rvert = \lvert x \rvert = \lvert -\delta/2 \rvert = \delta/2 < \delta \) but \( \lvert f(x) - f(a) \rvert = \lvert f(x) - f(0) \rvert = \lvert f(x) - 1 \rvert = \lvert f(-\delta/2) - 1 \rvert = 1 - 1 = 0 < 1 = \varepsilon \).

4. **Let \( f \) be defined as follows:**

   \[
   f(x) = \begin{cases} 
   0 & \text{if } x \text{ is rational} \\
   1 & \text{otherwise}
   \end{cases}
   \]

   **Is \( f \) continuous at some point? Justify your answer.** (5 pts.)

   **Proof:** No, \( f \) is not continuous anywhere. Indeed let \( a \in \mathbb{R} \) be any real number. Let \( \varepsilon = 1 \) (or any positive real number less than 1). Let \( \delta > 0 \) be any. Choose \( x \in (a - \delta, a + \delta) \) such that \( x \) is rational if \( a \) is irrational and \( x \) is irrational if \( a \) is rational. (Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), there is such an \( x \)). Now \( \lvert f(x) - f(a) \rvert = 1 = \varepsilon \geq \varepsilon \). Thus \( f \) is not continuous at \( a \).

5. **Show that if \( f \) and \( g \) are continuous, then so is their sum \( f + g \).** (6 pts.)
Proof. Let $a \in \mathbb{R}$. Let $\varepsilon > 0$. Since $f$ is continuous, there is a $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta_1$ then $|f(x) - f(a)| < \varepsilon/2$. Similarly, since $g$ is continuous, there is a $\delta_2 > 0$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta_1$ then $|g(x) - g(a)| < \varepsilon/2$. Now let $\delta = \min(\delta_1, \delta_2)$. For any $x \in \mathbb{R}$ such that $|x - a| < \delta$, we have, $|(f+g)(x) - (f+g)(a)| = |f(x) + g(x) - (f(a) + g(a))| \leq |f(x) - f(a)| + |g(x) + g(a)| \leq |f(x) - f(a)| + |g(x) + g(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

6. Show that if $f$ and $g$ are continuous, then so is their product $f \cdot g$. (7 pts.)

Proof. Let $a \in \mathbb{R}$. Let $\varepsilon > 0$.

Since $f$ is continuous at $a$, there is a $\delta_1$ such that if $|x - a| < \delta_1$ then $|f(x) - f(a)| < 1$,
i.e. $f(a) - 1 < f(x) < f(a) + 1$. Let $M = \max(|f(a) - 1|, |f(a) + 1|)$. Then for all $x \in \mathbb{R}$ for which $|x - a| < \delta_2$, we have $|f(x)| < M$.

Let $N$ be such that $|g(a)| < N$.

Since $f$ is continuous at $a$ there is a $\delta_2 > 0$ such that for all $x \in \mathbb{R}$ for which $|x - a| < \delta_2$, we have $|f(x) - f(a)| < \varepsilon/2N$.

Since $g$ is continuous at $a$ there is a $\delta_3 > 0$ such that for all $x \in \mathbb{R}$ for which $|x - a| < \delta_3$, we have $|g(x) - g(a)| < \varepsilon/2M$.

Now let $\delta = \max(\delta_1, \delta_2, \delta_3)$. Then for all $x \in \mathbb{R}$ for which $|x - a| < \delta$, we have:

$$\left| |fg(x)| - |fg(a)| \right| = |f(x)g(x) - f(a)g(a)| = |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \leq |f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| = |f(x)||g(x) - g(a)| + |f(x) - f(a)||g(a)| < M(\varepsilon/2M) + (\varepsilon/2N)N = \varepsilon.$$

7. By applying the previous questions show that the function defined by

$$f(x) = x^2 - 4x + \sqrt{2}$$

is continuous. (5 pts.)

Proof: The identity function is continuous by Q2. The squaring function $g(x) = x^2$ is continuous from this and Q6. By multiplying the constant $-4$ function with Id, by Q1 and Q6, we see that $h(x) = -4x$ is continuous as well. The constant function $k(x) = \sqrt{2}$ being also continuous, by Q5, $f = g + h + k$ is continuous as well.

8. By using directly the definition of continuity show that the function defined by

$$f(x) = x^2 - 4x + \sqrt{2}$$

is continuous. (7 pts.)

Proof: Let $a \in \mathbb{R}, \varepsilon > 0, M = \max(|a - 1|, |a + 1|)$ and $\delta = \min(1, \varepsilon/(M + a + 4))$.

Then if $x \in \mathbb{R}$ is such that $|x - a| < \delta$, then $-\delta < x - a < \delta$ and $a - 1 \leq a - \delta < x < a + \delta \leq a + 1$, and so $|x| < M$. Hence

$$|x^2 - 4x + \sqrt{2} - (a^2 - 4a + \sqrt{2})| = |x^2 - a^2 - 4x + 4a - 4\sqrt{2}| = |(x - a)(x + a - 4)| < |x - a||x + a - 4| < |x - a||x + a + 4| < \varepsilon.'
10. Show that exp $x$ defined by $\sum_{n \geq 0} x^n/n!$ is continuous. (15 pts.)

**Proof.** We know that exp $x$ exists (i.e. the series converges for all $x$). Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Let $M = \max(|a - 1|, |a + 1|)$ and $\delta = \min(1, \varepsilon/\exp M)$. Now assume $x \in \mathbb{R}$ satisfies $|x - al| < \delta$. We have, first of all $|x| < M$ as in Q8. It is easy to show that $|a| < M$ as well. Finally,

$|\sum_{n \geq 0} x^n/n! - \sum_{n \geq 0} a^n/n!| = |\sum_{n \geq 0} (x^n - a^n)/n!| = |\sum_{n \geq 1} (x^n - a^n)/n!|$

$= \sum_{n \geq 1} |x^n - a^n|/n! = \sum_{n \geq 1} |x^n - a^n|/n!$

$\leq |x - al| \sum_{n \geq 1} nM^{n-1}/n! = |x - al| \sum_{n \geq 1} M^{n-1}/(n-1)!$

$= |x - al| \exp M < \varepsilon$

11. Let $f$ be continuous at $a$ and assume that $f(a) > 0$. Show that there is an interval around $a$ where $f$ is strictly positive. (15 pts.)

**Proof.** Since $f$ is continuous, by choosing $\varepsilon = f(a)/2 > 0$, we find a $\delta$ such that for $x \in (a - \delta, a + \delta)$, $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) = (f(a)/2, 3f(a)/2) \subseteq \mathbb{R}^>0$.

12. Let $f$ be continuous. Assume that $f(a) < 0$ and $f(b) > 0$. Show that $f(c) = 0$ for some $c$ between $a$ and $b$. (15 pts.)

13. Generalize the concept of a continuous function to metric spaces. (10 pts.)