Math 151 Final (1)

Ali Nesin
January, 2005

1. Discuss the convergence of the series

$$\sigma(s) = \sum_{n\geq 0} \frac{3n+1}{(4n+1)^s}$$

for $s > 0$. (10 pts.)

Answer: For $n > 0$, $(3n+1)/(4n+1)^s < (3n+1)/(4n)^s \leq 4n/(4n)^s = 1/(4n)^{s-1}$, so that, by the comparison test, $\sigma(s)$ converges for $s > 2$.

For $n > 0$, $(3n+1)/(4n+1)^s \geq n/(5n)^s \geq n/25n^s = 1/25n^{s-1}$, so that, by the comparison test, $\sigma(s)$ converges for $s \leq 2$.

Remark. Since $(3n + 1)/(4n + 1)^s = n(3 + 1/n)/n^s(4 + 1/n)^s \approx 3/4^s n^{s-1}$, the limit of the general term is not 0 if $s < 1$, so that in that case the series diverge.

2. Let $s = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + ...$ We know that s is a real number because the series converges. Now regroup the terms of the series as follows:

$$(1/1 - 1/2 - 1/4) + (1/3 - 1/6 - 1/8) + (1/5 - 1/10 - 1/12) + ...$$

i.e. as

$$\sum_{n\geq 0}\left(\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4}\right).$$

Show that this rearrangement/regroupment of s is equal to $s/2$. (15 pts.)

Proof: Let $s_n = 1 - 1/2 + ... + (-1)^n/(n+1)$ be the partial sums of s. By definition of the convergence of series $\lim_{n\to\infty} s_n = s$.

Let t_n be the partial sums of the second series and $t = \lim_{n\to\infty} t_n$ in case this limit exists. Since

$$\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} = \frac{1}{4n+2} - \frac{1}{4n+4} = \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right),$$

we have $t_n = s_{2n+2} / 2$. Therefore $t = \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_{2n+2} / 2 = s/2$.

Remark: The purpose of the question is to show that one cannot regroup the terms of a series without altering the sum, so any attempt to answer the question by grouping the terms of series is wrong.

3. Discuss the convergence and the divergence of the series (25 pts.)

3a. $\sum_{n\geq 1} \frac{n!(3n)!}{(4n)!} 9^n$,

3b. $\sum_{n\geq 1} (-1)^n (\sqrt{n+1} - \sqrt{n})$,

3c. $\sum_{n\geq 1}\left(\frac{1}{n^2} + \frac{3}{n}\right)$,

3d. $\sum_{n\geq 1} (n!)^{1/n}$,

3e. $\sum_{n\geq 1} \frac{n!}{n^n}$

Answers:

3a. We apply the ratio test:

$$\frac{(n+1)!(3n+3)!9^{n+1}}{(4n+4)!}\frac{(4n)!}{n!(3n)!9^{n}}=\frac{(n+1)(3n+1)(3n+2)(3n+3)9}{(4n+1)(4n+2)(4n+3)(4n+4)}\approx\frac{3^5}{4^4}=\frac{243}{256}<1;$$

thus the series converges.

3b. This is an alternating series of the form $\sum (-1)^n a_n$ with $a_n \geq 0$. Since,

$$\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\approx 0,$$

the general term converges to 0. It is enough to prove that $a_n \geq a_{n+1}$, which is quite easy.

Remark. Does the series converge absolutely? It would be wrong to reason as follows:

$$\sum_{n\geq 1}(\sqrt{n+1}-\sqrt{n})=(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\ldots=-\sqrt{1}=-1,$$

(after simplification!) and as you know the sum of positive numbers cannot be equal to -1. The partial sums of these series are $\sqrt{n}-1$, and this sequence diverges to infinity.

3c. We know $\sum 1/n^2$ converges, so that if the series converged, the difference, which is $3\sum 1/n$ would also converge, a contradiction

3d. $(n!)^{1/n} \geq 1$ for all $n \geq 1$. Thus the general term cannot converge to 0, therefore the series is divergent.

3e. We apply the ratio test:

$$\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}=\frac{n^n}{(n+1)^n}=\frac{1}{(1+1/n)^n}=\frac{1}{e}<1,$$

therefore the series converges.

4. Discuss the convergence and the divergence of the series (10 pts.)

$$\sigma(p,a)=\sum_{n\geq 1}n^p a^n$$

where $p \in \mathbb{Z}$ and $a \in \mathbb{R}$.

Solution. Assume $a \geq 0$ first. Then the n-th root of the general series converges to a. Hence the series converges absolutely if $|a| < 1$ and diverges if $|a| > 1$.

If $a = 1$ then the series converges if $p < -1$ and diverges otherwise.

If $a = -1$ then the series becomes an alternating series and it is easy to show that it converges if $p < 0$ and diverges otherwise.

5. Suppose $x_n > 0$ and $y_n > 0$ for every $n \in \mathbb{N}$. Suppose further that $\lim_{x\to\infty} x_n/y_n = 1$. Show that $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ either both converge or diverge. (20 pts.)

Proof: The situation being symmetric, we may assume that $\sum_{n\geq 1} y_n$ is convergent. By assumption there is an N such that for all $n > N$, $|x_n/y_n - 1| < 1$, i.e. $-1 < x_n/y_n - 1 < 1$ and so $x_n < 2y_n$. This is enough to prove the result.

6. Suppose that $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ are both convergent series with positive terms. Show that $\sum_{n\geq 1} x_n y_n$ converges. (20 pts.)

Proof: Let M be an upper bound for the sequence $(y_n)_n$. Then $\sum_{n\geq 1} x_n y_n \leq M \sum_{n\geq 1} x_n$ and therefore the series $\sum_{n\geq 1} x_n y_n$ whose terms are positive is bounded, and hence is convergent.

Remark. We have showed more: We have showed that if $\sum_{n\geq 1} x_n$ is convergent series with positive terms and $(y_n)_n$ is a bounded and positive sequence then $\sum_{n\geq 1} x_n y_n$ is convergent.