Algebra
(Field Theory and Local Fields)

Resit I

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Ali Nesin

p always denotes a prime in $\mathbb{N}$.

If k is a nonarchimedean valuation field (with values in $\mathbb{R}^*$), $\mathfrak{o} = \mathfrak{o}(k)$ denotes its valuation ring, $\wp = \wp(k)$ the unique maximal ideal of $\mathfrak{o}$.

If k is also complete with respect to the valuation $|\ \ |$ and K is an extension of degree n of k, recall that

$$|\alpha| = |N_{K/k}(\alpha)|^{1/n} = |\det(m_\alpha : K \to K)|^{1/n}$$

defines the unique (necessarily complete) valuation on K that extends the valuation on k. We let $f = [\mathfrak{o}(K)/\wp(K) : \mathfrak{o}(k)/\wp(k)]$ and $e = [K^* : k^*]$. Recall that the extension K/k is called **unramified** if $f = n$ and **completely ramified** if $f = 1$.

1. What is the characteristic of $\mathbb{Q}_p$? (1 or –99 pts.)

2. Show that in a finite field F, $\sum_{a\in \mathbf{F}} a^m = 0$ if $p - 1$ does not divide m. (10 pts.)

3. Let A be an $n \times n$ matrix over a commutative ring such that $A^m = 0$ for some $m \in \mathbb{N}$. Show that Id + A is invertible. (7 or –4 pts.)

4. Let $q \in \mathbb{Q}$. Define $|q|_6 = 6^{-k}$ if $q = 6^k a/b$ where $a, b \in \mathbb{Z}$, $b \neq 0$ and 6 divides neither a nor b. Is $|\ \ |_6$ a norm on $\mathbb{Q}$? (5 or –95 pts.)

5. Show that $\sum_{n\in\mathbf{N}} np^n$ converges in $\mathbb{Q}_p$. Show that it is it is in fact in $\mathbb{Q}$. (10 pts.)

6a. Show that $\sum_{n>0} \frac{1}{n} p^n$ converges in $\mathbb{Q}_p$. (15 pts.)

6b*. Show that the above element is not the root of any nonzero polynomial over $\mathbb{Q}$. (30 pts.)

7. For $n \in \mathbf{N}$, let $n^* \in \{0, 1, \ldots, p-1\}$ be such that $n \equiv n^* \pmod p$. Show that $\sum_{n\in\mathbf{N}} n^* p^n$ converges in $\mathbb{Q}_p$ and is in $\mathbb{Q}$. (**Hint:** Find an explicit formula). (10 pts.)

8. Let k be a field of characteristic $\neq 2$. Let K be an extension of degree 2 of k.

8a. Show that $K = k[\delta]$ for some $\delta \in K \setminus k$ such that $d := \delta^2 \in k^*$. (5 pts.)

8b. Show that there is a one-to-one correspondance between the set of extensions of degree ≤ 2 of k and $k^*/(k^*)^2$. (7 pts.)

From now on we assume that k is a complete field with respect to a nonarchimedean valuation $| \ |$. We let $K = k[\delta]$ and $d = \delta^2 \in k^*$ as in part a.

8c. Show that the unique valuation on $k[\delta]$ is given by

$$|a + b\delta| = |a^2 - b^2 d|^{1/2}.$$

(From now on d and δ are as in part a). (3 pts.)

8d. Show that we may choose d so that $d \in \mathrm{o}(k)$. (3 pts.)

8e. Conclude that $\mathrm{o}(k)[\delta] \leq \mathrm{o}(k[\delta])$ and that $\wp(k)[\delta] \leq \wp(k[\delta])$. (5 pts.)

9. We assume $p \neq 2$ and we continue with the above exercise.

9a. Show that p is not a square in $\mathbb{Q}_p$. (4 pts.)

9b. Show that there is an element u in $\mathbb{Z}_p^*$ which is not a square in $\mathbb{Q}_p$. (In fact we may choose $u \in \{2, ..., p-1\}$). (5 or -10 pts.)

9c. Show that up is not a square in $\mathbb{Q}_p$. (4 or -5 pts.)

9d. Show that $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (5 pts.)

9e. Conclude that $\mathbb{Q}_p$ has 3 extensions of degree 2 where d can be chosen to be one of u, p, up. (2 pts.)

9f. Show that if $d = p$ or up, then $e = 2$. (10 pts.)

9g. Show that if $d = p$ or up, then

$$\mathrm{o}(\mathbb{Q}_p[\delta]) = \mathbb{Z}_p[\delta]$$

$$\wp(\mathbb{Q}_p[\delta]) = \{a + b\delta : a \in p\mathbb{Z}_p \text{ and } b \in \mathbb{Z}_p\}.$$

Conclude that $d = p$ or up, then $f = 1$. (15 pts.)

9h. Show that if $d = u \in \mathbb{Z}_p^*$, then $f = 2$ and $e = 1$. (15 pts.)

10. Show that $\mathbb{Q}_2$ has exactly 7 nonisomorphic extensions of degree 2 given by $\mathbb{Q}_2[\delta]$ where $\delta^2 = -1, 2, -2, 5, -5, 10, -10$. Which of those extensions are unramified and which ones are completely ramified? (30 pts.)