Math 331

Resit Exam
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Do not use symbols like $\exists$, $\forall$, $\Rightarrow$. Make full and precise sentences.

To prove the questions 1-7 use only the well-known properties of $(\mathbb{R}, +, \times, <)$ which also hold in $\mathbb{Q}$ together with the:

Supremum Property of $\mathbb{R}$. *Every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound*[1].

1. *Show that the supremum property is false for $\mathbb{Q}$.*

First Answer. Let $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$. Since $0 \in A$, A is nonempty. Since any element of A is < 2, A is bounded above (e.g. by 2). On the other hand A has no least upper bound in $\mathbb{Q}$ because $\sqrt{2} \notin \mathbb{Q}$. (To show this you may need the Arcimedean Property which is the content of Question 3).

Second answer. Consider any increasing bounded sequence whose limit is not a rational number, e.g. the sequence 0.1, 0.12, 0.123,... This sequence has no least upper bound in $\mathbb{Q}$ because the upper bound of this sequence would be its limit and the limit, having nonperiodic digits, is not in $\mathbb{Q}$.

2. *Show that the least upper bound of a subset is unique when it exists.*

Answer. Let A be a nonempty bounded set. Let a and b be the least upper bounds. Since a is an upper bound and since a is a least upper bound, $a \leq b$. Similarly $b \leq a$. Thus $a = b$.

3. *Prove the* **Archimedean Property**, *namely that for any $r \in \mathbb{R}$, there is a natural number $n \in \mathbb{N}$ such that $r < n$.*

Answer. Assume not. Then the set $\mathbb{N}$ of natural numbers is bounded above by r. Therefore $\mathbb{N}$ has a least upper bound. Let b be the least upper bound of $\mathbf{N}$. Thus $n \leq b$ for all $n \in \mathbb{N}$. Since for a natural number n, $n + 1$ is also a natural number, it follows that $n + 1 \leq b$ for all natural numbers n. Then $n \leq b - 1$ for all $n \in \mathbb{N}$ and $b - 1$ is also an upper bound for $\mathbf{N}$. Since $b - 1 < b$, this contradicts the fact that b is the **least** upper bound.

4. *Show that any nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.*

Answer. Let A be a subset of $\mathbb{R}$ which is bounded below. Then the set $-A$ is bounded above. Let b be the least upper bound of $-A$. Then $-b$ is the greatest lower bound of A as it can be checked easily.

[1] An element b is called the least upper bound of a subset X of $\mathbb{R}$ if b is an upper bound for X and if b is less than any upper bound for X.

5. *Show that for any* $\varepsilon > 0$, *there is an* $n \in \mathbb{N}$ *such that* $1/n < \varepsilon$.
Answer. Apply Question 3 to $r = 1/\varepsilon$.

6. *Show that if* $r \in \mathbb{R}$, *there is an* $n \in \mathbb{Z}$ *such that* $n - 1 \le r \le n$.
Answer. By Question 3, there are integers greater than r. By Question 4, there is a least integer n greater than r. Then $n - 1$ is less than r.

7. *Show that there is a rational number between any two distinct real numbers.*
Answer. Let $r < s$ be two real numbers. By Question 5 there is an integer n such that $1/n < s - r$. Consider the set $\{m \in \mathbb{N} : m < nr\}$. Then $(m + 1)/n$ is a rational number between r and s.

8. *By using the definition of limits, show that* $\lim_{n\to\infty} (-1)^n$ *does not exist.*
Answer. Let $\varepsilon = 1$ and N **any** natural number. Let $n > N$ be even and $m > N$ be odd. Then $|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \varepsilon$.

9. *By using the definition of limits, show that* $\lim_{n\to\infty} \dfrac{2n}{n+1} = 2$.
Answer. Let $\varepsilon > 0$ be any real number. Let N be any natural number $> 2/\varepsilon - 1$. Then for any $n > N$, $\left|\dfrac{2n}{n+1} - 2\right| < \varepsilon$ as it can be checked easily.

10. *Show that if* $a_n \le a_{n+1} \le b_{n+1} \le b_n$ *for all n, then* $\bigcap_{n\in N} [a_n, b_n]$ *is a nonempty interval.*
It is easy to show that the intersection of convex sets is convex, so that $\bigcap_{n\in N} [a_n, b_n]$ is a convex subset of **R**, hence is an interval. Let us show that it is nonempty. The sequence $(a_n)_n$ is bounded above by each of the b_n. Let a be the least upper bound of this sequence. Then $a \ge a_n$ because a is an upper bound of the a_n. Also $a \le b_n$ because a is the **least** upper bound. Hence a is in $\bigcap_{n\in N} [a_n, b_n]$. Similarly if b is the greatest lower bound of the sequence $(b_n)_n$, then b is in $\bigcap_{n\in N} [a_n, b_n]$. In fact, an easy argument shows that $\bigcap_{n\in N} [a_n, b_n] = [a, b]$.

11. *Show that a Cauchy sequence is bounded.*
Let $(x_n)_n$ be a Cauchy sequence. Let $\varepsilon = 1$. Let N be such that for $n, m \ge N$, $|x_n - x_m| < \varepsilon = 1$. Let $\alpha = \max(1, |x_0 - x_N|, \dots, |x_{N-1} - x_N|)$. Then $|x_n - x_N| \le \alpha$ for every n. Thus the sequence is bounded.

12. *Let* $x_1 = 1$, $x_2 = 2$ *and* $x_n = (x_{n-1} + x_{n-2})/2$ *for* $n > 2$.
12a. *Show that* $1 \le x_n \le 2$ *for all n.*
By induction on n. For $n = 1$ and 2 this is true by hypothesis. Assuming x_{n-1} and x_{n-2} are between 1 and 2, the definition of x_n yields immediately that x_n is between 1 and 2.
12b. *Show that* $|x_n - x_{n+1}| = 1/2^{n-1}$ *for all n.*
By induction on n. The equality clearly holds for $n = 1$. For $n \ge 1$ we compute by using the inductive assumption and the definition: $|x_{n+1} - x_{n+2}| = |x_{n+1} - (x_{n+1} + x_n)/2| = |(x_{n+1} - x_n)/2| = 1/2^n$.

12c. *Show that if* $m > n$ *then* $|x_n - x_m| < 1/2^{n-2}$ *for all* n.
We compute using the previous question. $|x_n - x_m| \leq |x_n - x_{n+1}| + ... + |x_{m-1} - x_m| = 1/2^{n-1} + ... + 1/2^{m-2} = 1/2^{n-1}(1 + ... + 1/2^{m-n-1}) = 1/2^{n-1}(1 - 1/2^{m-n})2 = 1/2^{n-2} - 1/2^{m-2} < 1/2^{n-2}$.
12d. *Show that* $(x_n)_n$ *is a Cauchy sequence.*
Follows immediately from the previous question.
12e. *Find its limit.*
Note that the sequence is formed by taking the arithmetic mean of the previous two terms. It can be guessed that the limit is at the two thirds of the way from $x_1 = 1$ to $x_2 = 2$, i.e. it is 5/3. This can be shown by some elementary linear algebra. But there is an easier way: Note that question 12b can be sharpened into $x_n - x_{n+1} = (-1)^n/2^{n-1}$, thus $x_{n+1} - 1 = x_{n+1} - x_1 = (x_{n+1} - x_n) + ... + (x_2 - x_1) = 1 - 1/2 + 1/4 - ... + (-1)^{n-1}/2^{n-1}$. It follows that the limit x, that we know it exists from the previous question, is equal to the infinite sum $x = 1 + 1 - 1/2 + 1/4 - ... + (-1)^{n-1}/2^{n-1} + ...$ Now just remarque that $2x = 2 + 2 - 1 + 1/2 - 1/4 + ... = 3 + (1/2 - 1/4 + 1/8 - ...) = 3 + (2 - x)$ and so $x = 5/3$.

13. *We say that a sequence* $(x_n)_n$ *is* ***contractive*** *if there is a constant* c, $0 < c < 1$, *such that* $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$ *for all* n. *Show that every contractive sequence is convergent.*
Answer. Note first that $|x_{n+1} - x_n| \leq c^n|x_1 - x_0|$. Thus we can estimate $|x_n - x_m|$ as is done above and show that such a sequence is Cauchy.