Analysis (Math 162)
Final

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Do not use symbols such as $\Rightarrow$, $\forall$.
Make full sentences.
Write legibly. Use correct punctuation.
Explain your ideas.

I. Convergent Sequences. For each of the topological spaces (X, τ), describe the convergent sequences and discuss the uniqueness of their limits.

1. $\tau = \wp(X)$. ($\wp(X)$ is the set of all subsets of X).
2. $\tau = \{\emptyset, X\}$.
3. $a \in X$ is a fixed element and τ is the set of all subsets of X that do not contain a, together with X of course.
4. $a \in X$ is a fixed element and τ is the set of all subsets of X that contain a, together with $\emptyset$ of course.
5. τ is the set of all cofinite subsets of X.

II. Subgroup Topology on $\mathbb{Z}$. Let $\tau = \{n\mathbb{Z}+m : n,\, m \in \mathbb{Z},\, n \neq 0\} \cup \{\emptyset\}$. We know that $(\mathbb{Z}, \tau)$ is a topological space.

1. Let $a \in \mathbb{Z}$. Is $\mathbb{Z} \setminus \{a\}$ open in τ?
2. Find an infinite non open subset of $\mathbb{Z}$.
3. Let $a,\, b \in \mathbb{Z}$. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f_{a,b}(z) = az + b$ continuous? (Prove or disprove).
4. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(z) = z^2$ continuous? (Prove or disprove).
5. Is the topological space $(\mathbb{Z}, \tau)$ compact? (Prove or disprove).

III. Miscellaneous.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the squaring map. Suppose that the arrival set is endowed with the usual Euclidean topology. Find the smallest topology on the domain that makes f continuous.

2. Let τ be the topology on $\mathbb{R}$ generated by $\{[a, b) : a, b \in \mathbb{R}\}$. Compare this topology with the Euclidean topology. Is this topology generated by a metric?

3. Show that the series $\sum_{i=0}^{n} x^n/n!$ converges for any $x \in \mathbb{R}$. Show that the map $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\exp(x) = \sum_{i=0}^{n} x^n/n!$ is continuous.