Math 152 Midterm

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i. Decide the convergence of the series

\[ \sum_n \frac{1}{\sqrt{|n^2 - 2|}} \]

**Answer:** Since for all \( n > 1 \),

\[ \frac{1}{\sqrt{|n^2 - 2|}} = \frac{1}{\sqrt{n^2 - 2}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n} \]

and since \( \sum_n \frac{1}{n} \) diverges, the series \( \sum_n \frac{1}{\sqrt{|n^2 - 2|}} \) diverges as well.

ii. Decide the convergence of the series

\[ \sum_n \frac{1}{\sqrt{n^2 + 1}} \]

**Answer:** Since for \( n > 0 \),

\[ \frac{1}{\sqrt{n^2 + 1}} \geq \frac{1}{\sqrt{n^2 + n^2}} = \frac{1}{n\sqrt{2}} \]

and since \( \sum_n \frac{1}{n} \) diverges, the series \( \sum_n \frac{1}{\sqrt{n^2 + 1}} \) diverges as well.

iii. Decide the convergence of the series

\[ \sum_n \frac{1}{\sqrt{|n^4 - 6|}} \]

**Answer:** Since for \( n > 1 \),

\[ \frac{1}{\sqrt{|n^4 - 6|}} = \frac{1}{\sqrt{n^4 - 6}} \leq \frac{1}{\sqrt{n^4 - n^4/2}} = \frac{\sqrt{2}}{n^2} \]

and since \( \sum_n \frac{1}{n^2} \) converges, the series \( \sum_n \frac{1}{\sqrt{|n^4 - 6|}} \) converges as well.
iv. Suppose that the series $\sum_n a_n$ is convergent. Show that $\lim_{n \to \infty} a_n = 0$.

**Proof:** Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n \to \infty} s_n = s$. We have $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1}) = s - s = 0$.

v. Suppose that $(a_n)_n$ is a positive and decreasing sequence and that the series $\sum_n a_n$ is convergent. Show that $\lim_{n \to \infty} na_n = 0$.

**Proof:** We know that $\lim_{n \to \infty} a_n = 0$. Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n \to \infty} s_n = s$. Since $(a_n)_n$ is decreasing,

$$na_{2n} \leq a_{n+1} + \ldots + a_{2n} = s_{2n} - s_n.$$

Thus $\lim_{n \to \infty} na_{2n} = \lim_{n \to \infty} (s_{2n} - s_n) = s - s = 0$. Hence $\lim_{n \to \infty} 2na_{2n} = 0$.

Also $0 < (2n + 1)a_{2n+1} \leq (2n + 1)a_{2n} = 2na_{2n} + a_{2n}$. By the above and the fact that $\lim_{n \to \infty} a_{2n} = 0$, the right hand side converges to 0. Hence by the squeezing lemma $\lim_{n \to \infty} (2n + 1)a_{2n+1} = 0$.

From the above two paragraphs it follows that $\lim_{n \to \infty} na_n = 0$.

(Remark: As we will see later, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. This example shows that the conditions that $(a_n)_n$ is positive and decreasing and that $\lim_{n \to \infty} na_n = 0$ are not enough for the series $\sum_n a_n$ to be convergent.)

vi. Find a positive sequence $(a_n)_n$ such that the series $\sum_n a_n$ is convergent but that $\lim_{n \to \infty} na_n \neq 0$.

**Solution:** Take $a_n = 1/n^2$ if $n$ is not a square and $a_n = 1/n$ if $n$ is a square. Then $(na_n)_n$ does not converge as $\lim_{n \to \infty} (n^2 + 1)a_{n^2 + 1} = 0$ and $\lim_{n \to \infty} (2n^2 + 2n + 2a_n + 2n) = 1$. On the other hand $\sum_n a_n = \sum_n$ non square $a_n + \sum_n$ a square $a_n = \sum_n$ non square $1/n^2 + \sum_n 1/n^2 < 2\sum_n 1/n^2 < 4$.

vii. Suppose that series $\sum_n a_n$ is absolutely convergent and that the sequence $(b_n)_n$ is Cauchy. Show that the series $\sum_n a_n b_n$ is absolutely convergent.

**Proof:** Since $(b_n)_n$ is Cauchy the sequence $(|b_n|)_n$ is bounded. In fact that is all we need to conclude. Indeed, let $B$ be an upper bound of the sequence $(|b_n|)_n$. Then $|a_n b_n| \leq B|a_n|$ and since $\sum_n |a_n|$ converges, $\sum_n |a_n b_n|$ converges as well.

viii. Let $(a_n)_n$ be a sequence. Suppose that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. Such a sequence is called of **bounded variation**. Show that a sequence of bounded variation converges.

**Proof:** Since $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges, $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ converges as well. Thus the sequence of partial sums whose terms are

$$\sum_{i=1}^{n-1} (a_i - a_{i+1}) = a_1 - a_n$$
converges, say to $a$. Thus $(a_n)_n$ converges to $a_1 - a$. 