Math 152 Midterm

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April 2004

i. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{|n^2-2|}}.$$

Answer: Since for all $n > 1$,
$$\frac{1}{\sqrt{|n^2-2|}} = \frac{1}{\sqrt{n^2-2}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$
and since $\sum_n \frac{1}{n}$ diverges, the series $\sum_n \frac{1}{\sqrt{|n^2-2|}}$ diverges as well.

ii. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{n^2+1}}.$$

Answer: Since for $n > 0$,
$$\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{n\sqrt{2}}$$
and since $\sum_n \frac{1}{n}$ diverges, the series $\sum_n \frac{1}{\sqrt{n^2+1}}$ diverges as well.

iii. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{|n^4-6|}}.$$

Answer: Since for $n > 1$,
$$\frac{1}{\sqrt{|n^4-6|}} = \frac{1}{\sqrt{n^4-6}} \leq \frac{1}{\sqrt{n^4-n^4/2}} = \frac{\sqrt{2}}{n^2}$$
and since $\sum_n \frac{1}{n^2}$ converges, the series $\sum_n \frac{1}{\sqrt{n^4-6}}$ converges as well.

iv. Suppose that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} a_n = 0$.

Proof: Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n\to\infty} s_n = s$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$.

v. Suppose that $(a_n)_n$ is a positive and decreasing sequence and that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} na_n = 0$.

Proof: We know that $\lim_{n\to\infty} a_n = 0$. Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n\to\infty} s_n = s$.

Since $(a_n)_n$ is decreasing,

$$na_{2n} \leq a_{n+1} + \ldots + a_{2n} = s_{2n} - s_n.$$

Thus $\lim_{n\to\infty} na_{2n} = \lim_{n\to\infty}(s_{2n} - s_n) = s - s = 0$. Hence $\lim_{n\to\infty} 2na_{2n} = 0$.

Also $0 < (2n+1)a_{2n+1} \leq (2n+1)a_{2n} = 2na_{2n} + a_{2n}$. By the above and the fact that $\lim_{n\to\infty} a_{2n} = 0$, the right hand side converges to 0. Hence by the squeezing lemma $\lim_{n\to\infty}(2n+1)a_{2n+1} = 0$.

From the above two paragraphs it follows that $\lim_{n\to\infty} na_n = 0$.

(Remark: As we will see later, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. This example shows that the conditions that $(a_n)_n$ is positive and decreasing and that $\lim_{n\to\infty} na_n = 0$ are not enough for the series $\sum_n a_n$ to be convergent.)

vi. Find a positive sequence $(a_n)_n$ such that the series $\sum_n a_n$ is convergent but that $\lim_{n\to\infty} na_n \neq 0$.

Solution: Take $a_n = 1/n^2$ if n is not a square and $a_n = 1/n$ if n is a square. Then $(na_n)_n$ does not converge as $\lim_{n\to\infty}(n^2+1)a_{n^2+1} = 0$ and $\lim_{n\to\infty} n^2 a_{n^2} = 1$. On the other hand $\sum_n a_n = \sum_{n \text{ non square}} a_n + \sum_{n \text{ a square}} a_n = \sum_{n \text{ non square}} 1/n^2 + \sum_n 1/n^2 < 2\sum_n 1/n^2 < 4$.

vii. Suppose that series $\sum_n a_n$ is absolutely convergent and that the sequence $(b_n)_n$ is Cauchy. Show that the series $\sum_n a_n b_n$ is absolutely convergent.

Proof: Since $(b_n)_n$ is Cauchy the sequence $(|b_n|)_n$ is bounded. In fact that is all we need to conclude. Indeed, let B be an upper bound of the sequence $(|b_n|)_n$. Then $|a_n b_n| \leq B|a_n|$ and since $\sum_n |a_n|$ converges, $\sum_n |a_n b_n|$ converges as well.

viii. Let $(a_n)_n$ be a sequence. Suppose that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. Such a sequence is called of **bounded variation**. Show that a sequence of bounded variation converges.

Proof: Since $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges, $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges as well. Thus the sequence of partial sums whose terms are

$$\sum_{i=1}^{n-1}(a_i - a_{i+1}) = a_1 - a_n$$

converges, say to a. Thus $(a_n)_n$ converges to $a_1 - a$.