Nonarchimedian Discrete Valuations

Summer Midterm II
15th of June, 1999
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Prelude: Let $|\ \ |$ be a nonarchemedian discrete valuation on the field k. Let o be the ring of integers and $\wp$ the (unique) maximal ideal of o. Let $p \in \wp$ be a generator of $\wp$. Recall that $\mathrm{o} \setminus \wp = \mathrm{o}^*$.

1. Show that $(p^n\mathrm{o}^*)_{n \in \mathbf{N}}$ is a disjoint family of subsets of o. (2 pts.)

2. Show that $\mathrm{o} = \bigcup_{n=0}^{\infty} p^n \mathrm{o}^* \cup \{0\}$ (5 pts.)

3. Show that $\bigcap_{n=0}^{\infty} p^n \mathrm{o} = \{0\}$. (3 pts.)

4. Show that every nonzero ideal of o is of the form $p^n\mathrm{o}$ for some $n \in \mathbf{N}$. Thus o is a pid. (15 pts.)

5. Show that the valuation group $G = |k^*|$ is generated by $|p|$ and so is isomorphic to $\mathbf{Z}$. (15 pts.)

A series $\sum_{i=0}^{\infty} a_i$ is said to converge to s if the sequence $\sum_{i=0}^{n} a_i$ of partial sums converges to s.

6. Show that if k is complete, the series $\sum_{i=0}^{\infty} a_i$ converges iff the sequence $(a_n)_{n \in \mathbf{N}}$ converges to 0. (15 pts.)

7. Assume that the series $\sum_{i=0}^{\infty} a_i$ converges in k. Show that $\left|\sum_{i=0}^{\infty} a_i\right| \leq \max\{|a_n| : n \in \mathbf{N}\}$. (10 pts.)

8. Let $p(T)$ be an irreducible polynomial in $k[T]$. Let $(f_n(T))_{n \in \mathbf{N}}$ be a sequence of formal power series (i.e. $f_n(T) \in k[[T]]$ for each n). Show that $\sum_{n=0}^{\infty} f_n(T)p(T)^n$ converges in $k[[T]]$ for the $p(T)$-adic valuation. (5 pts.)

9. Assume k is complete. Let $A \subset \mathrm{o}$ be a set of representatives of $\mathrm{o}/\wp$. Show that every $a \in \mathrm{o}$ can be written uniquely as $\sum_{n=0}^{\infty} a_n p^n$ ($a_n \in A$). Conversely, show that such a series converges always. (15 + 5 pts.)

10. Let $k = \mathbf{Q}_p$ be the completion of $\mathbf{Q}$ for the p-adic valuation (p a prime). Show that the set A of the question above can be chosen to be $\{0, 1, ..., p - 1\}$. (10 pts.)