Nonarchimedean Discrete Valuations
Summer Midterm II
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Prelude: Let \( | \cdot | \) be a nonarchemedian discrete valuation on the field \( k \). Let \( o \) be the ring of integers and \( \mathfrak{o} \) the (unique) maximal ideal of \( o \). Let \( p \in \mathfrak{o} \) be a generator of \( \mathfrak{o} \). Recall that \( o \setminus \mathfrak{o} = o^* \).

1. Show that \((p^n o^*)_{n \in \mathbb{N}}\) is a disjoint family of subsets of \( o \). (2 pts.)

2. Show that \( o = \bigcup_{n=0}^{\infty} p^n o^* \cup \{0\} \) (5 pts.)

3. Show that \( \bigcap_{n=0}^{\infty} p^n o = \{0\} \). (3 pts.)

4. Show that every nonzero ideal of \( o \) is of the form \( p^n o \) for some \( n \in \mathbb{N} \). Thus \( o \) is a pid. (15 pts.)

5. Show that the valuation group \( G = | k^* | \) is generated by \( | p | \) and so is isomorphic to \( \mathbb{Z} \). (15 pts.)

A series \( \sum_{i=0}^{\infty} a_i \) is said to converge to \( s \) if the sequence \( \sum_{i=0}^{n} a_i \) of partial sums converges to \( s \).

6. Show that if \( k \) is complete, the series \( \sum_{i=0}^{\infty} d_i \) converges iff the sequence \( (a_n)_{n \in \mathbb{N}} \) converges to 0. (15 pts.)

7. Assume that the series \( \sum_{i=0}^{\infty} a_i \) converges in \( k \). Show that \( \left| \sum_{i=0}^{\infty} a_i \right| \leq \max \{ \left| a_n \right| : n \in \mathbb{N} \} \). (10 pts.)

8. Let \( p(T) \) be an irreducible polynomial in \( k[T] \). Let \((f_n(T))_{n \in \mathbb{N}}\) be a sequence of formal power series (i.e. \( f_n(T) \in k[[T]] \) for each \( n \)). Show that \( \sum_{n=0}^{\infty} f_n(T) p(T)^n \) converges in \( k[[T]] \) for the \( p(T) \)-adic valuation. (5 pts.)

9. Assume \( k \) is complete. Let \( A \subset o \) be a set of representatives of \( o/\mathfrak{o} \). Show that every \( a \in o \) can be written uniquely as \( \sum_{n=0}^{\infty} a_n p^n \) (\( a_n \in A \)). Conversely, show that such a series converges always. (15 + 5 pts.)

10. Let \( k = \mathbb{Q}_p \) be the completion of \( \mathbb{Q} \) for the \( p \)-adic valuation (\( p \) a prime). Show that the set \( A \) of the question above can be chosen to be \( \{0, 1, \ldots, p-1\} \). (10 pts.)