Analysis I (Math 121)
Resit – Correction

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1. Find a sequence neither decreasing nor increasing that converges to 1. (2 pts.)

   Answer: Let $a_n = 1 + \frac{(-1)^n}{n}$. It is clear that $\lim_{n\to\infty} a_n = 1$. Since the subsequence $(a_{2n})_n$ is decreasing and converges to 1 and the subsequence $(a_{2n})_n$ is increasing and converges to 1, the sequence (a_n) is neither increasing nor decreasing.

2. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $a_n \in \mathbb{Z}$ for all n. Is it true that $\lim_{n\to\infty} a_n \in \mathbb{Z}$? (4 pts.)

   Answer: Yes, it is true. In fact this is true even for Cauchy sequences: A Cauchy sequence $(a_n)_n$ whose terms are in $\mathbb{Z}$ is eventually constant, i.e. there is an N such that $a_n = a_N$ for all $n \geq N$, and this implies of course that $\lim_{n\to\infty} a_n = a_N \in \mathbb{Z}$. So, let us show that the Cauchy sequence $(a_n)_n$ is eventually constant.

   In the definition of Cauchy sequences, take $\epsilon = 1/2$. Thus, there is an M such that for all n, $m > M$, $|a_n - a_m| < 1/2$. But since a_n and a_m are in $\mathbb{Z}$, this means that for all n, $m > M$, $|a_n - a_m| = 0$, i.e. that $a_n = a_m$. Now take $N = M + 1$.

3. Let $(q_n)_n$ be a convergent sequence of real numbers. Suppose that $q_n \in \mathbb{Q}$ for all n. Is it true that $\lim_{n\to\infty} q_n \in \mathbb{Q}$? (3 pts.)

   Answer: Of course not! In fact every real number is the limit of a rational sequence. Indeed, let $r \in \mathbb{R}$. Let $n \in \mathbb{N} \setminus \{0\}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number $q_n \in (r - 1/n, r)$. Since $r - 1/n < q_n < r$, by the Sandwich Lemma, $\lim_{n\to\infty} q_n = r$.

4. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $5a_n/2 \in \mathbb{N}$ for all n. What can you say about $\lim_{n\to\infty} a_n$? (4 pts.)

   Answer: Let $\lim_{n\to\infty} a_n = r$. Then $\lim_{n\to\infty} 5a_n/2 = 5r/2$. By hypothesis and by part 2, $5r/2 \in \mathbb{Z}$. Thus $r = 2n/5$ for some $n \in \mathbb{N}$.

5. Let $(a_n)_n$ be a sequence of real numbers such that the subsequence $(a_{2n})_n$ converges. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

Answer: Of course not! We can have $a_{2n} = 1/n$ and $a_{2n+1} = n$.

6. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 1. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

   Answer: Of course not! We can have $a_n = (-1)^n$. Then $(a_n)_n$ is a sequence of alternating ones and minus ones, so that it diverges. And since $a_n^2 = 1$, the sequence $(a_n^2)_n$ converges to 1.

7. Let $(a_n)_n$ be a sequence of real numbers such that the subsequences $(a_{2n})_n$ and $(a_{2n+1})_n$ both converge. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

   Answer: Of course not! We can have $a_n = (-1)^n$. Then $(a_n)_n$ is a sequence of alternating ones and minus ones, so that it diverges. And since $a_{2n} = 1$ and $a_{2n+1} = -1$, the sequence $(a_{2n})_n$ converges to 1 and the sequence $(a_{2n+1})_n$ converges to -1.

8. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 0. Does the sequence $(a_n)_n$ converge necessarily? (8 pts.)

   Answer: Yes! Let $\epsilon > 0$. Let $\nu = \sqrt{\epsilon}$. Since the sequence $(a_n^2)_n$ converges to 0, there is an N such that for all $n > N$, $|a_n^2| < \nu$, i.e. $|a_n|^2 < \epsilon^2$. Since $|a_n|$ and ν are positive, this implies that $|a_n| < \epsilon$. Thus there is an N such that for all $n > N$, $|a_n| < \epsilon$; i.e. the sequence $(a_n)_n$ converges to 0.

9. Let $(a_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Is it true that $\lim_{n\to} a_{2n} = \infty$? (3 pts.)

   Answer: Yes! Let A be any real number. $\lim_{n\to\infty} a_n = \infty$, there is an N such that for all $n > N$, $a_n > A$. Then for $2n > N$, $a_{2n} > A$.

10. Assume $\lim_{n\to\infty} a_n$ exists and $a_n \neq 0$ for all n. Does the sequence $(a_{2n}/a_{2n+1})_n$ converge necessarily? (5 pts.)

    Answer: No, the sequence $(a_{2n}/a_{2n+1})_n$ may not converge if $\lim_{n\to\infty} a_n = 0$. For example, choose

    $$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

    Clearly $\lim_{n\to\infty} a_n = 0$, but

    $$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

    And the subsequence $\frac{(n+1)^2}{n}$ diverges to ∞, although the subsequence $\frac{n+1}{n^2}$ converges to 0.

    On the other hand, if the limit of the sequence $(a_n)_n$ is nonzero, say ℓ, then the sequence $(a_{2n}/a_{2n+1})_n$ converges to 1 because $\lim_{n\to\infty} a_{2n}/a_{2n+1} = \lim_{n\to\infty} a_{2n}/\lim_{n\to\infty} a_{2n+1} = \ell/\ell = 1$. Note that the last part uses the fact that ℓ is nonzero.

11. Find the following limits and prove your result using **only** the definition. (30 pts.)

a. $\lim_{n\to\infty} \frac{3n+105}{5n-79}$

Answer: $\lim_{n\to\infty} \frac{3n+105}{5n-79} = \frac{3}{5}$.

Proof: Let $\epsilon > 0$. Let N_1 be such that $32 < \epsilon N_1$. Let $N = \max(N_1, 395)$. Now for $n > N$, we have,

$$\left|\frac{3n+105}{5n-79} - \frac{3}{5}\right| = \left|\frac{762}{25n-395}\right| = \frac{762}{25n-395} \leq \frac{762}{24n} < \frac{32}{n} < \frac{32}{N_1} < \epsilon.$$

The first equality is simple computation. The second equality follows from the fact $n > N \geq 395 > 16$ (so that $25n - 395 > 0$). The third inequality follows from the fact that $n > N \geq 395$, so that $25n - 395 \geq 25n - n = 24n$.The fourth inequality is also a simple computation.

b. $\lim_{n\to\infty} \frac{n^2-5n+3}{-100n+2}$

Answer: $\lim_{n\to\infty} \frac{n^2-5n+3}{-100n+2} = -\infty$.

Proof: It is enough to show that $\lim_{n\to\infty} \frac{n^2-5n+3}{100n-2} = \infty$.

We first note that the two roots of $n^2 - 5n + 3$ are $\frac{5\pm\sqrt{25-12}}{2} = \frac{5\pm\sqrt{13}}{2}$, so that if $n \geq 5 > \frac{9}{2} = \frac{5+\sqrt{16}}{2} > \frac{5+\sqrt{13}}{2}$, then $n^2 - 5n + 3 > 0$.

Now let $A \in \mathbb{R}$ be any real number. Let $N = \max(100A + 5, 5)$. Now, for all $n > N$,

$$\frac{n^2-5n+3}{100n-2} > \frac{n^2-5n+3}{100n} > \frac{n^2-5n}{100n} = \frac{n-5}{100} > \frac{N-5}{100} = A.$$

Here, the first inequality follows from the fact that $n > N \geq 5$, so that $n^2 - 5n + 3 > 0$.

c. $\lim_{n\to\infty} \frac{n-8}{2n^3-89}$.

Answer: $\lim_{n\to\infty} \frac{n-8}{2n^3-89} = 0$.

Proof: Let $\epsilon > 0$. Let $N = \max(1/\epsilon, 89)$. Now for $n > N$, $\left|\frac{n-8}{2n^3-89}\right| = \frac{n-8}{2n^3-89} < \frac{n}{2n^3-n} = \frac{1}{2n^2-1} < \frac{1}{n^2} < \frac{1}{n} < \epsilon$.

12. Find (16 pts. Justify your answers).

a. $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n}$.

Answer: $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = 0$.

Proof: We use the fact that $2/3 < 1$. Since $\lim_{n\to\infty} \frac{6n}{n^2+1} = 0$, there is an N such that for all $n > N$, $\frac{6n}{n^2+1} < 1/6$. Then $0 \leq \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = (2/3+1/6)^{3n} = (5/6)^{3n}$. By Sandwich Lemma $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = 0$.

b. $\lim_{n\to\infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n}$.

Answer: $\lim_{n\to\infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} = \infty$.

Proof: We use the fact that $5/4 > 1$. Since $\lim_{n\to\infty} \frac{7}{n^5} = 0$, there is an N such that for all $n > N$, $\frac{7}{n^5} < 1/8$. Then $\left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} > (5/4 - 1/8)^{n^n} = (9/8)^{n^n} \geq (9/8)^n$. Since $(9/8) > 1$, $\lim_{n\to\infty} (9/8)^n = \infty$. The result follows.

13. Find $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts.).

Answer: $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

Proof: Since $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right) = 0$, there is an N_1 such that for all $n > N_1$, $\frac{n^2-1}{n^3-n-5} < 1/2$. On the other hand, for $n > 3$, $\frac{n^2-1}{2n-3} < \frac{n^2-1}{n} < n$. Let $N = \max(3, N_1)$. Now for $n > N$, $\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} < (1/2)^{\frac{n^2-1}{2n-3}} < (1/2)^n$. Since the right hand side converges to 0, by Sandwich Lemma, $0 \leq \lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$. (For the first inequality, one needs the fact that $n^3 - n - 5 > 0$ for $n \geq 2$. This follows from the facts that $2^3 - 2 - 5 = 1 > 0$ and $n^3 - n - 5 < (n+1)^3 - (n+1) - 5$. And this last inequality is easy to show).

14. Show that the series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3}$ converges. Find an upper bound for the sum. (10 pts.)

Answer: $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3} = \sum_{n=1}^{\infty} (1/n)^{n/3} < \sum_{n=1}^{\infty} 1/2^{n/3} =$
$\sum_{n=1}^{\infty} 1/2^{3n/3} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+1}{3}} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+2}{3}}$
$= \sum_{n=1}^{\infty} 1/2^n + \frac{1}{2^{1/3}} \sum_{n=0}^{\infty} 1/2^n + \frac{1}{2^{1/3}} \sum_{n=0}^{\infty} 1/2^n = 1/2 + 2^{-1/3} + 2^{-2/3} < 5$.

15. Let $(a_n)_n$ be a sequence of real numbers. Assume that there is an $r > 1$ such that $|a_{n+1}| \geq r|a_n|$ for all n. What can you say about the convergence or the divergence of $(a_n)_n$? (6 pts.)

Answer: The sequence diverges. Furthermore the sequence diverges to ∞ if it is eventually positive and to $-\infty$ if it is eventually negative.

Proof: One can show by induction on n that $|a_n| > r^n|a_0|$. Thus $\lim_{n\to} |a_n| = \infty$ (because $r > 1$). It should now be clear that the answer is valid.