1. Find a sequence neither decreasing nor increasing that converges to 1. (2 pts.)

**Answer:** Let \( a_n = 1 + \frac{(-1)^n}{n} \). It is clear that \( \lim_{n \to \infty} a_n = 1 \). Since the subsequence \( (a_{2n})_n \) is decreasing and converges to 1 and the subsequence \( (a_{2n})_n \) is increasing and converges to 1, the sequence \( (a_n) \) is neither increasing nor decreasing.

2. Let \( (a_n)_n \) be a convergent sequence of real numbers. Suppose that \( a_n \in \mathbb{Z} \) for all \( n \). Is it true that \( \lim_{n \to \infty} a_n \in \mathbb{Z} \)? (4 pts.)

**Answer:** Yes, it is true. In fact this is true even for Cauchy sequences: A Cauchy sequence \( (a_n)_n \) whose terms are in \( \mathbb{Z} \) is eventually constant, i.e. there is an \( N \) such that \( a_n = a_N \) for all \( n \geq N \), and this implies of course that \( \lim_{n \to \infty} a_n = a_N \in \mathbb{Z} \). So, let us show that the Cauchy sequence \( (a_n)_n \) is eventually constant.

In the definition of Cauchy sequences, take \( \epsilon = 1/2 \). Thus, there is an \( M \) such that for all \( n, m > M \), \( |a_n - a_m| < 1/2 \). But since \( a_n \) and \( a_m \) are in \( \mathbb{Z} \), this means that for all \( n, m > M \), \( |a_n - a_m| = 0 \), i.e. that \( a_n = a_m \). Now take \( N = M + 1 \).

3. Let \( (q_n)_n \) be a convergent sequence of real numbers. Suppose that \( q_n \in \mathbb{Q} \) for all \( n \). Is it true that \( \lim_{n \to \infty} q_n \in \mathbb{Q} \)? (3 pts.)

**Answer:** Of course not! In fact every real number is the limit of a rational sequence. Indeed, let \( r \in \mathbb{R} \). Let \( n \in \mathbb{N} \setminus \{0\} \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), there is a rational number \( q_n \in (r - 1/n, r) \). Since \( r - 1/n < q_n < r \), by the Sandwich Lemma, \( \lim_{n \to \infty} q_n = r \).

4. Let \( (a_n)_n \) be a convergent sequence of real numbers. Suppose that \( 5a_n/2 \in \mathbb{N} \) for all \( n \). What can you say about \( \lim_{n \to \infty} a_n \)? (4 pts.)

**Answer:** Let \( \lim_{n \to \infty} a_n = r \). Then \( \lim_{n \to \infty} 5a_n/2 = 5r/2 \). By hypothesis and by part 2, \( 5r/2 \in \mathbb{Z} \). Thus \( r = 2n/5 \) for some \( n \in \mathbb{N} \).

5. Let \( (a_n)_n \) be a sequence of real numbers such that the subsequence \( (a_{2n})_n \) converges. Does the sequence \( (a_n)_n \) converge necessarily? (2 pts.)
6. Let \((a_n)_n\) be a sequence of real numbers such that the sequence \((a_n^2)_n\) converges to 1. Does the sequence \((a_n)_n\) converge necessarily? (2 pts.)

**Answer:** Of course not! We can have \(a_{2n} = 1/n\) and \(a_{2n+1} = n\).

7. Let \((a_n)_n\) be a sequence of real numbers such that the subsequences \((a_{2n})_n\) and \((a_{2n+1})_n\) both converge. Does the sequence \((a_n)_n\) converge necessarily? (2 pts.)

**Answer:** Of course not! We can have \(a_n = (-1)^n\). Then \((a_n)_n\) is a sequence of alternating ones and minus ones, so that it diverges. And since \(a_n^2 = 1\), the sequence \((a_n^2)_n\) converges to 1.

8. Let \((a_n)_n\) be a sequence of real numbers such that the sequence \((a_n^2)_n\) converges to 0. Does the sequence \((a_n)_n\) converge necessarily? (8 pts.)

**Answer:** Yes! Let \(\epsilon > 0\). Let \(\nu = \sqrt{\epsilon}\). Since the sequence \((a_n^2)_n\) converges to 0, there is an \(N\) such that for all \(n > N\), \(|a_n^2| < \nu\), i.e. \(|a_n|^2 < \epsilon\). Since \(|a_n|\) and \(\nu\) are positive, this implies that \(|a_n| < \epsilon\). Thus there is an \(N\) such that for all \(n > N\), \(|a_n| < \epsilon\); i.e. the sequence \((a_n)_n\) converges to 0.

9. Let \((a_n)_n\) be a sequence of real numbers such that \(\lim_{n \to \infty} a_n = \infty\). Is it true that \(\lim_{n \to \infty} a_{2n} = \infty\)? (3 pts.)

**Answer:** Yes! Let \(A\) be any real number. \(\lim_{n \to \infty} a_n = \infty\), there is an \(N\) such that for all \(n > N\), \(a_n > A\). Then for \(2n > N\), \(a_{2n} > A\).

10. Assume \(\lim_{n \to \infty} a_n = \ell\) exists and \(a_n \neq 0\) for all \(n\). Does the sequence \((a_{2n}/a_{2n+1})_n\) converge necessarily? (5 pts.)

**Answer:** No, the sequence \((a_{2n}/a_{2n+1})_n\) may not converge if \(\lim_{n \to \infty} a_n = 0\). For example, choose

\[
a_n = \begin{cases} 
1/n & \text{if } n \text{ is even} \\
1/n^2 & \text{if } n \text{ is odd}
\end{cases}
\]

Clearly \(\lim_{n \to \infty} a_n = 0\), but

\[
a_n/a_{n+1} = \begin{cases} 
(n+1)^2/n^2 & \text{if } n \text{ is even} \\
(n+1)/n^2 & \text{if } n \text{ is odd}
\end{cases}
\]

And the subsequence \((n+1)^2/n^2\) diverges to \(\infty\), although the subsequence \((n+1)/n^2\) converges to 0.

On the other hand, if the limit of the sequence \((a_n)_n\) is nonzero, say \(\ell\), then the sequence \((a_{2n}/a_{2n+1})_n\) converges to 1 because \(\lim_{n \to \infty} a_{2n}/a_{2n+1} = \lim_{n \to \infty} a_{2n}/\lim_{n \to \infty} a_{2n+1} = \ell/\ell = 1\). Note that the last part uses the fact that \(\ell\) is nonzero.
11. Find the following limits and prove your result using only the definition. (30 pts.)

a. \( \lim_{n \to \infty} \frac{3n + 105}{5n - 79} \)

**Answer:** \( \lim_{n \to \infty} \frac{3n + 105}{5n - 79} = \frac{3}{5} \).

**Proof:** Let \( \epsilon > 0 \). Let \( N_1 \) be such that \( 32 < \epsilon N_1 \). Let \( N = \max(N_1, 395) \). Now for \( n > N \), we have,

\[
\left| \frac{3n + 105}{5n - 79} - \frac{3}{5} \right| = \frac{762}{25n - 395} \leq \frac{32}{24} < \frac{32}{N_1} < \epsilon.
\]

The first equality is simple computation. The second equality follows from the fact \( n > N \geq 395 > 16 \) (so that \( 25n - 395 > 0 \)). The third inequality follows from the fact that \( n > N \geq 395 \), so that \( 25n - 395 \geq 25n - n = 24n \). The fourth inequality is also a simple computation.

b. \( \lim_{n \to \infty} \frac{n^2 - 5n + 3}{100n + 2} \)

**Answer:** \( \lim_{n \to \infty} \frac{n^2 - 5n + 3}{100n + 2} = -\infty \).

**Proof:** It is enough to show that \( \lim_{n \to \infty} \frac{n^2 - 5n + 3}{100n + 2} = \infty \).

We first note that the two roots of \( n^2 - 5n + 3 \) are \( \frac{5 + \sqrt{25 - 12}}{2} = \frac{5 + \sqrt{13}}{2} \), so that if \( n \geq \frac{9}{2} \), then \( n^2 - 5n + 3 > 0 \).

Now let \( A \in \mathbb{R} \) be any real number. Let \( N = \max(100A + 5, 5) \). Now, for all \( n > N \),

\[
\frac{n^2 - 5n + 3}{100n + 2} > \frac{n^2 - 5n + 3}{100n} = \frac{n - 5}{100} > \frac{N - 5}{100} = A.
\]

Here, the first inequality follows from the fact that \( n > N \geq 5 \), so that \( n^2 - 5n + 3 > 0 \).

c. \( \lim_{n \to \infty} \frac{n^8}{2n^3 - 89} \)

**Answer:** \( \lim_{n \to \infty} \frac{n^8}{2n^3 - 89} = 0 \).

**Proof:** Let \( \epsilon > 0 \). Let \( N = \max(1/\epsilon, 89) \). Now for \( n > N \),

\[
\left| \frac{n^8}{2n^3 - 89} \right| = \frac{n^8}{2n^3 - 89} < \frac{n}{2n^3 - n} = \frac{1}{2n^2 - 1} < \frac{1}{n^2} < \frac{1}{n} < \epsilon.
\]


a. \( \lim_{n \to \infty} \left( \frac{2}{3} + \frac{6n}{n^3 + 1} \right)^{3n} \)

**Answer:** \( \lim_{n \to \infty} \left( \frac{2}{3} + \frac{6n}{n^3 + 1} \right)^{3n} = 0 \).

**Proof:** We use the fact that \( 2/3 < 1 \). Since \( \lim_{n \to \infty} \frac{6n}{n^3 + 1} = 0 \), there is an \( N \) such that for all \( n > N \), \( \frac{6n}{n^3 + 1} < 1/6 \). Then \( 0 \leq \left( \frac{2}{3} + \frac{6n}{n^3 + 1} \right)^{3n} = (2/3 + 1/6)^{3n} = (5/6)^{3n} \). By Sandwich Lemma \( \lim_{n \to \infty} \left( \frac{2}{3} + \frac{6n}{n^3 + 1} \right)^{3n} = 0 \).
b. \( \lim_{n \to \infty} \left( \frac{2}{3} - \frac{2}{n} \right)^n \).

**Answer:** \( \lim_{n \to \infty} \left( \frac{2}{3} - \frac{2}{n} \right)^n = \infty. \)

**Proof:** We use the fact that \( 5/4 > 1 \). Since \( \lim_{n \to \infty} \frac{2}{n} = 0 \), there is an \( N \) such that for all \( n > N \), \( \frac{2}{n} < 1/8 \). Then \( \left( \frac{2}{3} - \frac{2}{n} \right)^n > (5/4 - 1/8)n^n = (9/8)^n \geq (9/8)^n \). Since \( (9/8) > 1 \), \( \lim_{n \to \infty} (9/8)^n = \infty. \) The result follows.

13. Find \( \lim_{n \to \infty} \left( \frac{n^2 - 1}{n^3 - n - 5} \right)^{\frac{n^2-1}{n^2+3}} \). (10 pts.)

**Answer:** \( \lim_{n \to \infty} \left( \frac{n^2 - 1}{n^3 - n - 5} \right)^{\frac{n^2-1}{n^2+3}} = 0. \)

**Proof:** Since \( \lim_{n \to \infty} \left( \frac{n^2 - 1}{n^3 - n - 5} \right) = 0 \), there is an \( N_1 \) such that for all \( n > N_1 \), \( \frac{n^2 - 1}{n^3 - n - 5} < 1/2 \). On the other hand, for \( n > 3 \), \( \frac{n^2 - 1}{2n - 3} < \frac{2^2 - 1}{n} < n. \)

Let \( N = \max(3, N_1) \). Now for \( n > N \), \( \left( \frac{n^2 - 1}{n^3 - n - 5} \right)^{\frac{n^2-1}{n^2+3}} < (1/2)^{\frac{n^2-1}{n^2+3}} < (1/2)^n. \) Since the right hand side converges to 0, by Sandwich Lemma, \( 0 \leq \lim_{n \to \infty} \left( \frac{n^2 - 1}{n^3 - n - 5} \right)^{\frac{n^2-1}{n^2+3}} = 0. \) (For the first inequality, one needs the fact that \( n^3 - n - 5 > 0 \) for \( n \geq 2 \). This follows from the facts that \( 2^3 - 2 - 5 = 1 > 0 \) and \( n^3 - n - 5 < (n + 1)^3 - (n + 1) - 5. \) And this last inequality is easy to show).

14. Show that the series \( \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n/3} \) converges. Find an upper bound for the sum. (10 pts.)

**Answer:** \( \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n/3} = \sum_{n=1}^{\infty} (1/n)^{n/3} < \sum_{n=1}^{\infty} 1/2^{n/3} = \sum_{n=1}^{\infty} 1/2^{n/3} + \sum_{n=0}^{\infty} 1/2^{3n+1} + \sum_{n=0}^{\infty} 1/2^{3n+2} = \sum_{n=1}^{\infty} 1/2^n + 1/27 \sum_{n=0}^{\infty} 1/2^n + 1/81 \sum_{n=0}^{\infty} 1/2^n = 1/2 + 2^{-1/3} + 2^{-2/3} < 5. \)

15. Let \( (a_n) \) be a sequence of real numbers. Assume that there is an \( r > 1 \) such that \( |a_{n+1}| \geq r|a_n| \) for all \( n \). What can you say about the convergence or the divergence of \( (a_n) \)? (6 pts.)

**Answer:** The sequence diverges. Furthermore the sequence diverges to \( \infty \) if it is eventually positive and to \(-\infty \) if it is eventually negative.

**Proof:** One can show by induction on \( n \) that \( |a_n| > r^n |a_0| \). Thus \( \lim_{n \to \infty} |a_n| = \infty \) (because \( r > 1 \)). It should now be clear that the answer is valid.