Analysis I (Math 121)
Final

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Justify all your answers. A nonjustified answer will not receive any grade whatsoever, even if the answer is correct. DO NOT use symbols such as $\forall$, $\exists$, $\Rightarrow$. Make full sentences with correct punctuation. You may write in Turkish or in English.

1. *Let $(a_n)_n$ be a convergent sequence of real numbers.*

a. *Does the sequence $(a_{2n})_n$ converge necessarily?* (2 pts.)

Since $(a_{2n})_n$ is a subsequence of the converging sequence $(a_n)_n$, both sequences converge to the same limit.

Remarks. We have seen in class that a subsequence of a conc-verging sequence converges.

Contrary to what some of you think, $a_{2n} \neq 2a_n$!

b. *Assume $a_n \neq 0$ for all n. Does the sequence $(a_n/a_{n+1})_n$ converge necessarily?* (2 pts.)

No, the sequence $(a_n/a_{n+1})_n$ may not converge if $\lim_{n\to\infty} a_n = 0$. For example, choose

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\lim_{n\to\infty} a_n = 0$, but

$$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

And the subsequence $\frac{(n+1)^2}{n}$ diverges to ∞, although the subsequence $\frac{n+1}{n^2}$ converges to 0.

On the other hand, if the limit of the sequence $(a_n)_n$ is nonzero, say ℓ, then the sequence $(a_n/a_{n+1})_n$ converges to 1 because $\lim_{n\to\infty} a_n/a_{n+1} = \lim_{n\to\infty} a_n / \lim_{n\to\infty} a_{n+1} = \ell/\ell = 1$. Note that the last part uses the fact that ℓ is nonzero.

2. *Find the following limits and prove your result using only the definition.* (30 pts.)

a. $\lim_{n\to\infty} \frac{2n-5}{5n+2}$.

We claim that the limit is $2/5$. Let $\epsilon > 0$. Since $\mathbb{R}$ is Archimedean, there is an N such that $2 < \epsilon N$. Now for $n > N$, $|\frac{2n-5}{5n+2} - \frac{2}{5}| = \frac{29}{5(5n+2)} < \frac{29}{25n} < 2/n < 2/N < \epsilon$.

b. $\lim_{n\to\infty} \frac{2n^2-5}{-5n+2}$.

We claim that the limit is $-\infty$. For this it is enough to prove that $\lim_{n\to\infty} \frac{2n^2-5}{5n-2} = \infty$. Let A be any real number. Let $N = \max(5A, 2)$. For $n > N$ we have $\frac{2n^2-5}{5n-2} > \frac{2n^2-5}{5n} > \frac{2n^2-n^2}{5n} = \frac{n^2}{5n} = \frac{n}{5} > \frac{N}{5} \geq A$. This proves that $\lim_{n\to\infty} \frac{2n^2-5}{5n-2} = \infty$.

c. $\lim_{n\to\infty} \frac{2n^2-5}{n^3+2}$.

We claim that the limit is 0. Let $\epsilon > 0$. Let N_1 be such that $2 > \epsilon N_1$ (Archimedean property of $\mathbb{R}$). Let $N = \max(1, N)$. Then for all $n > N$,

$$\left|\frac{2n^2-5}{n^3+2}\right| = \frac{2n^2-5}{n^3+2} < \frac{2n^2}{n^3+2} \leq \frac{2n^2}{n^3} = \frac{2}{n} < \frac{2}{N} < \epsilon.$$

Note that the first equality is valid because $n^3 - 5 \geq 0$, the second inequality is valid because $n^3 + 2 > 0$.

3. *Find* (16 pts. Justify your answers).

a. $\lim_{n\to\infty}(1/2 + 1/n)^n$.

Note that for $n \geq 3$, $0 < 1/2 + 1/n \leq 1/2 + 1/3 = 5/6$. Thus the sequence $((1/2+1/n)^n)_n$ is eventually squeezed between the zero constant sequence and the sequence $((5/6)^n)_n$. Since $\lim_{n\to\infty}(5/6)^n = 0$ (because $5/6 < 1$), $\lim_{n\to\infty}(1/2 + 1/n)^n = 0$.

b. $\lim_{n\to\infty}(3/2 - 7/n)^n$.

Since $3/2 > 1$ and $\lim_{n\longrightarrow\infty} 7/n = 0$, there is an N such that $7/N < 1/2 = 3/2 - 1$. In fact, it is enough to take $N = 15$. Then for all $n \geq N$, $3/2 - 7/n \geq 3/2 - 7/N > 1$ and so $(3/2-7/n)^n \geq (3/2-7/N)^n$. Therefore the sequence $((3/2-7/n)^n)_n$ is greater than the sequence $((3/2-7/N)^n)_n$. Since $3/2 - 7/N > 1$, the sequence $((3/2 - 7/N)^n)_n$ diverge to ∞. Hence $\lim_{n\to\infty}(3/2 - 7/n)^n = \infty$.

4. *Find* $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts. Justify your answer).

Assume $n > 5$. Then we have, $n^3 - n - 5 > n^3 - 2n > n^3 - n^3/2 = n^3/2 > 0$. Therefore,

$$\left|\frac{n^2-1}{n^3-n-5}\right| = \frac{n^2-1}{n^3-n-5} < \frac{n^2}{n^3-n-5} < \frac{n^2}{n^3/2} = 2/n.$$

Also $\frac{n^2-1}{2n-3} > \frac{n^2-1}{2n} > \frac{n^2-n}{2n} = \frac{n-1}{2} > 2$. Hence

$$\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} < (2/n)^2.$$

It follows that $\lim_{n\to\infty}\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

5. *Show that the series* $\sum_{n=1}^{\infty}(1/n)^n$ *converges. Find an upper bound for the sum.* (10 pts.)

   Since for $n \geq 2$, $1/n \leq 1/2$, we have $\sum_{n=1}^{\infty}(1/n)^n \leq 1 + \sum_{n=2}^{\infty}(1/n)^n \leq 1 + \sum_{n=1}^{\infty}(1/2)^n = 1 + \frac{1}{2}\sum_{n=0}^{\infty}(1/2)^n = 1 + 1/2 = 3/2$.

6. *Let* $(a_n)_n$ *be a sequence of nonnegative real numbers. Suppose that the sequence* $(a_n^2)_n$ *converges to* a. *Show that the sequence* $(a_n)_n$ *converges to* $\sqrt{a}$. (15 pts.)

   Note first that, since $a_n \geq 0$, $a \geq 0$ as well.

   Let $\epsilon > 0$.

   Case 1: $a > 0$**.** Since $\lim_{n\longrightarrow\infty} a_n^2 = a$, there is an N_2 such that for all $n > N$, $|a_n^2 - a| < \epsilon a$. Now for all $n > N$, $|a_n - \sqrt{a}| = \frac{|a_n^2-a|}{a_n+a} \leq \frac{|a_n^2-a|}{a} < \epsilon$.

   Case 2. $a = 0$**.**

   Since the sequence $(a_n^2)_n$ converges to 0, there is an N such that for all $n > N$, $a_n^2 < \epsilon^2$. So $(\epsilon + a_n)(\epsilon - a_n) = \epsilon^2 - a_n^2 > 0$. Since $\epsilon > 0$ and $a_n \geq 0$, we can divide both sides by $\epsilon + a_n$ to get $\epsilon - a_n > 0$, i.e. $a_n < \epsilon$. Since $a_n \geq 0$, this implies $|a_n| < \epsilon$.

7. *We have seen in class that the sequence given by* $a_n = ((1 + 1/n)^n)_n$ *converges to a real number* > 1. *Let* e *be this limit. Do the following sequences converge? If so find their limit.* (15 pts. Justify your answers).

   a) $\lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^n$.

   $\lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}1+\frac{1}{n+1}} = \frac{e}{1} = e$. The first equality is algebraic. The second equality holds because the limits of the numerator and the denominator exist and they are nonzero. The third equality holds because $\left(\left(1 + \frac{1}{n+1}\right)^{n+1}\right)_n$ is a subsequence of $\left(\left(1 + \frac{1}{n}\right)^n\right)_n$.

   b) $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{3n}$.

   $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{3n} = \lim_{n\to\infty}\left(\left(1 + \frac{1}{2n}\right)^{2n}\right)^{3/2} = e^{3/2}$ by Question 6.

   c) $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n^2}$.

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right)^{n} = \lim_{n\longrightarrow\infty} e^{n} = \infty$ because $e > 1$.