Analysis I (Math 121)
First Midterm
Correction

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Ali Nesin

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1. Which of the following are not vector spaces over $\mathbb{R}$ (with the componentwise addition and scalar multiplication) and why?

$$\begin{array}{l} V_1 = \{(x, y, z) \in \mathbb{R}^3 : xy \geq 0\} \\ V_2 = \{(x, y, z) \in \mathbb{R}^3 : 3x - 2y + z = 0\} \\ V_3 = \{(x, y, z) \in \mathbb{R}^3 : xyz \in \mathbb{Q}\} \\ V_4 = \{(x, y) \in \mathbb{R}^3 : x + y \geq 0\} \\ V_5 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} \\ V_6 = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\} \end{array}$$

(2+2+2+2+2+5 pts.)

Answers. V_1 is not a vector space because e.g. $(-1, 0, 0) \in V_1$, $(0, 1, 0) \in V_1$ but their sum $(-1, 1, 0) \notin V_1$.

V_2 is a vector space.

V_3 is not a vector space because e.g. $(1, 1, 1) \in V_3$, but $\sqrt{2}(1, 1, 1) \notin V_3$.

V_4 is not a vector space because e.g. $(1, 2, 1) \in V_4$, but $-(1, 2, 1) \notin V_4$.

V_5 is a vector space because $V_5 = \{(0, 0, 0)\}$.

V_6 is not a vector space because e.g. $(1, i) \in V_6$, $(1, -i) \in V_6$, but their sum $(2, 0) \notin V_6$.

2. On the set $X = \{2, 3 \ldots, 100\}$ define the relation $x \prec y$ by "$x \neq y$ and x divides y".

a) Show that this defines a partial order on X. (3 pts.)

b) Is this a linear order? (2 pts.)

c) Find all the maximal and minimal elements of this poset. (5 pts.)

Answers. By definition, we have,

$$2 \prec 4 \prec 8 \prec 16 \prec \dots$$
$$2 \prec 6 \prec 12 \prec 24 \prec 48 \prec \dots$$
$$3 \prec 6 \prec 12 \prec \dots$$
$$3 \not\prec 4 \not\prec 6$$

a) Yes, this is a partial order: Clearly $x \not\prec x$ for any x. Since division is transitive, $\prec$ is transitive as well.

b) No, because e.g. 2 and 3 are not comparable.

c) The prime numbers are minimal elements. The maximal elements are the numbers which are greater than 50. (For example 53 and 97 are both minimal and maximal. 47 is minimal but not maximal, since $47 \prec 94$. 94 is maximal but not minimal. 50 is neither minimal nor maximal.)

3. On $\mathbb{R} \times \mathbb{R}$ define the relation $\prec$ as follows $(x, y) \prec (x_1, y_1)$ by "either $y < y_1$, or $y = y_1$ and $x < x_1$".

   a) Show that this is a linear order. (5 pts.)

   b) Does every subset of this linear order which has an upper bound has a least upper bound? (5 pts.)

   Answers. a) **Irreflexivity.** Clearly $(x, y) \not\prec (x, y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

   b) **Transitivity.** Assume $(x, y) \prec (x_1, y_1)$ and $(x_1, y_1) \prec (x_2, y_2)$. We must have $y \leq y_1$ and $y_1 \leq y_2$. If either $y < y_1$ or $y_1 < y_2$, then $y < y_2$ and so $(x, y) \prec (x_2, y_2)$. Assume $y = y_1 = y_2$. Then $x < x_1 < x_2$, so $x < x_2$. Hence $(x, y) \prec (x_2, y_2)$.

   c **Comparability.** Let (x, y) and (x_1, y_1) be two elements of $\mathbb{R} \times \mathbb{R}$. We want to show that these two elements are comparable with respect to the partial order $\prec$. If $y < y_1$ or $y_1 < y$, then these two elements are comparable. Assume $y = y_1$. If $x < x_1$ or $x_1 < x$, then these two elements are comparable. The only case left is the case where these two elements are equal.

   Thus the relation $\prec$ is a linear order.

   b) No. For example the set $\mathbb{R} \times \{0\}$ is bounded above by $(0, 1)$ but it does not have a least upper bound.

4. For each $n \in \mathbb{N}$, let a_n and b_n be two real numbers. Assume that for each n, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. Show that $\cap_{n \in \mathbb{N}}[a_n, b_n] = [a, b]$ for some real numbers a and b. (10 pts.)

   Proof: Since the set $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0, it has a least upper bound, say a. Similarly the set $\{b_n : n \in \mathbb{N}\}$ has a greatest lower bound, say b. I claim that $\cap_{n \in \mathbb{N}}[a_n, b_n] = [a, b]$.

   If $x \geq a$, then $x \geq a_n$ for all n. Likewise, if $x \leq b$, then $x \leq b_n$ for all n. Hence, if $x \in [a, b]$, then $x \in [a_n, b_n]$ for all n.

Conversely, let $x \in \cap_{n\in\mathbb{N}}[a_n, b_n]$. Then $a_n \le x \le b_n$ for all n. Thus x is an upper bound for $\{a_n : n \in \mathbb{N}\}$ and a lower bound for $\{b_n : n \in \mathbb{N}\}$. Hence $a \le x \le b$.

5. Show that for any natural number n and for any real number $x \in [0, 1)$,

$$(1-x)^n \le 1 - nx + \frac{n(n-1)}{2}x^2.$$

(10 pts.)

Proof: We proceed by induction on n.

If $n = 0$, then both sides are equal to 1, and so the result holds.

Suppose we know the result for n, i.e. suppose we know that for any real number $x \in [0, 1)$,

$$(1-x)^n \le 1 - nx + \frac{n(n-1)}{2}x^2.$$

We will prove that for any real number $x \in [0, 1)$,

$$(1-x)^{n+1} \le 1 - (n+1)x + \frac{(n+1)n}{2}x^2.$$

Let $x \in [0, 1)$. Since $(1-x)^n \le 1 - nx + \frac{n(n-1)}{2}x^2$ and since $1 - x > 0$, multiplying by $1 - x$ both sides we get

$$\begin{array}{rcl} (1-x)^{n+1} & = & (1-x)^n(1-x) \\ & \le & (1 - nx + \frac{n(n-1)}{2}x^2)(1-x) \\ & = & 1 - (n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3. \end{array}$$

Thus

$$(1-x)^{n+1} \le 1 - (n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3.$$

Since $x \ge 0$,

$$\begin{array}{rcl} (1-x)^{n+1} & \le & 1 - (n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3 \\ & \le & 1 - (n+1)x + \frac{(n+1)n}{2}x^2. \end{array}$$

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6. a) Show that for any complex number α there is a polynomial of the form $p(X) = X^2 + aX + b \in \mathbb{R}[X]$ such that $p(\alpha) = 0$. (Note: a and b should be real numbers). (10 pts.)

b) What can you say about a and b if $\alpha = u + iv$ for some $u, v \in \mathbb{Z}$? (5 pts.)

Proof: a) Let α be a complex number. Then $p(x) := (x-\alpha)(x-\overline{\alpha}) = x^2 - (\alpha+\overline{\alpha})x + \alpha\overline{\alpha} \in \mathbb{R}[x]$ (because $\alpha + \overline{\alpha}$ and $\alpha\overline{\alpha}$ are real numbers) and it is easy to check that $p(\alpha) = 0$.

b) It is clear that if $\alpha = u + iv$ for some $u, v \in \mathbb{Z}$, then $\alpha + \overline{\alpha} = 2u$ and $\alpha\overline{\alpha} = u^2 + v^2$ are integers and so $p(x) \in \mathbb{Z}[x]$.

Second Proof of part a. Write $\alpha = u + iv$ where u and v are real numbers. Then $\alpha^2 = u^2 - v^2 + 2uvi$. Thus $\alpha^2 - 2u\alpha = (u^2 - v^2 + 2uvi) - 2u(u + iv) = -u^2 - v^2$, so that $\alpha^2 - 2u\alpha + (u^2 + v^2) = 0$. Hence α is a root of the polynomial $p(x) = x^2 - 2ux + (u^2 + v^2) \in \mathbb{R}[x]$.

Part b follows from this immediately.

7. a) Show that for any $\alpha \in \mathbb{C}$ there is a $\beta \in \mathbb{C}$ such that $\beta^2 = \alpha$. (15 pts.)

   b) Show that for any $\alpha, \beta \in \mathbb{C}$ there is an $x \in \mathbb{C}$ such that $x^2 + \alpha x + \beta = 0$. (10 pts.)

   Proof: a. Let $\alpha = a + bi$. We try to find $\beta \in \mathbb{C}$ such that $\beta^2 = \alpha$, i.e. we try to find two real numbers x and y such that $(x + iy)^2 = a + bi$. We may assume that $\alpha \neq 0$ (otherwise take $\beta = 0$). Thus we assume that a and b are noth both 0. After multiplying out, we see that this equation is equivalent to the system

$$\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}$$

   Since $x = 0$ implies $a = 0 = y = b$, which is contrary to our assumption, x must be nonzero. Thus we have $y = b/2x$ and so the above system is equivalent to the following:

$$\begin{aligned} x^2 - (b/2x)^2 &= a \\ y &= b/2x \end{aligned}$$

   Equalizing the denominators in the first one, we get the following equivalent system:

$$\begin{aligned} 4x^4 - 4ax^2 - b^2 &= 0 \\ y &= b/2x \end{aligned}$$

   So now the problem is about the solvability of the first equation $4x^4 - 4ax^2 - b^2 = 0$. (Once we find x, which is necessarily nonzero, we set $y = b/2x$). Setting $z = x^2$, we see that the solvability of $4x^4 - 4ax^2 - b^2 = 0$ is equivalent to the question of whether $4z^2 - 4az - b^2 = 0$ has a nonnegative solution. Since this last equation is a quadratic equation over $\mathbb{R}$, it is easy to answer this question. There are two possible solutions: $z = a \pm \sqrt{a^2 + b^2}$ and one of them $z = a + \sqrt{a^2 + b^2}$ is nonnegative (even if a is negative). Thus we can take

$$x = \sqrt{a + \sqrt{a^2 + b^2}}$$

   and

$$y = b/2x.$$

   b. We first compute as follows: $0 = x^2 + \alpha x + \beta = x^2 + \alpha x + \alpha^2/4 + (\beta - \alpha^2/4) = (x + \alpha/2)^2 + (\beta - \alpha^2/4)$. Thus a solution, x of this equation must satisfy $(x+\alpha/2)^2 = \alpha^2/4-\beta$. Hence if $z \in \mathbb{C}$ is such that $z^2 = \alpha^2/4-\beta$ (by the first part there is such a z), then $z-\alpha/2$ is a solution of $x^2+\alpha x+\beta = 0$. (The other solution is $-z - \alpha/2$).

8. Suppose X and Y are two subsets of $\mathbb{R}$ that have least upper bounds. Show that the set $X + Y := \{x + y : x \in X, y \in Y\}$ has a least upper bound and that $\sup(X + Y) = \sup(X) + \sup(Y)$. (15 pts.)

   Proof: Let a and b be the least upper bounds of X and Y respectively. Thus $x \le a$ for all $x \in X$ and $y \le b$ for all $y \in Y$. It follows that $x + y \le a + b$ for all $x \in X$ and $y \in Y$, meaning exactly that $a + b$ is an upper bound of $X + Y$. Now we show that $a + b$ is the least upper bound of $X + Y$. Let $\epsilon > 0$ be any. We need to show that $a + b - \epsilon < x + y$ for some $x \in X$ and $y \in Y$. Since a is the least upper bound of X, there is an $x \in X$ such that $a - \epsilon/2 < x$. Similarly there is a $y \in Y$ such that $b - \epsilon/2 < y$. Summing these two, we get $a + b - \epsilon < x + y$.

9. We consider the subset $X = \{1/2^n : n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ together with the usual metric, i.e. for $x, y \in X$, $d(x, y)$ is defined to be $|x - y|$. Show that the open subsets of X are the cofinite subsets[1] of X and the ones that do not contain 0. (20 pts.)

   Proof: We first show that the singleton set $\{1/2^n\}$ is an open subset of X. This is clear because $B(1/2^n, 1/2^{n+1}) = \{1/2^n\}$. It follows that any subset of X that does not contain 0 is open. Now let U be any cofinite subset of X. We proceed to show that U is open. If $0 \notin U$, then we are done by the preceding. Assume $0 \in U$. Since U is cofinite, there is a natural number $n_\circ$ such that for all $n > n_\circ$, $1/2^n \in U$, i.e. $B(0, 1/2^{n_\circ}) \subseteq U$. Now U is the union of $B(0, 1/2^{n_\circ})$ and of a finite subset not containing 0. Thus U is open.

   For the converse, we first show that a nonempty open ball is of the form described in the statement of the question. If the center of the ball is 0, then the ball is cofinite by the Archimedean property. If the center of the ball is not 0, then either the ball does not contain 0 or else it does contain 0, in which case the ball must be cofinite.

   To finish the proof, we must show that an arbitrary union of open balls each of which does not contain 0 cannot contain 0. But this is clear!

[1]A subset Y of X is called **cofinite** if $X \setminus Y$ is finite.