Math 331

Real and Complex Analysis
Midterm 1
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1a. Starting from the axioms for real numbers, show that for any real number $A > 0$ there is an integer $n > 0$ such that $A < n$. (Archimedean Property)

1b. Show that for any $\varepsilon > 0$ there is an integer $n > 0$ such that $1/n < \varepsilon$.

1c. By using the definition of limit of a sequence, show that $\lim_{n\to\infty} 1/n = 0$.

2a. Show that any increasing and bounded sequence of real numbers has a limit, namely the least upper bound of the sequence.

Let $(a_n)_n$ be a sequence of real numbers. We say that $\sum_{i=0}^{\infty} a_i$ converges to the real number ℓ if the sequence of "partial sums" $\left(\sum_{i=0}^{n} a_i\right)_n$ converges to ℓ. Let $r \in (0, 1)$ be a real number.

2b. Show that the sequence $\left(\sum_{i=0}^{n} r^n\right)_n$ is increasing and bounded by $\frac{1}{1-r}$.

2c. Show that $\sum_{i=0}^{\infty} r^n$ converges to the real number $\frac{1}{1-r}$.

2d. Let $(a_n)_n$ be a sequence of strictly positive real numbers. Show that if $a_{n+1}/a_n \le r$ for some $r \in (0, 1)$ then $\sum_{i=0}^{\infty} a_i$ converges.

2e. Let x be a positive real number. Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

2f. Let $(a_n)_n$ be a sequence of strictly positive real numbers. Assume that $\lim_{n\to\infty} a_{n+1}/a_n$ exists and is < 1. Show that $\sum_{i=0}^{\infty} a_i$ converges.

3. Let $f : [a, b] \to \mathbf{R}$ be continuous and increasing on a closed interval $[a, b]$. Let $A = f(a)$ and $B = f(b)$. Show that $f : [a, b] \to [A, B]$ is a bijection which has a continuous and increasing inverse.