Set Theory (Math 111)

April 9th, 2000

A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous at a point** $a \in \mathbb{R}$, if for all $\varepsilon > 0$ (real or rational, it does not matter) there is a $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous** if it is continuous at every point $a \in \mathbb{R}$.

1. Show that a constant function is continuous. (3 pts.)
2. Show that the identity function is continuous. (5 pts.)
3. Is the function f defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

continuous? Justify your answer. (6 pts.)
4. Let f be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise} \end{cases}$$

Is f continuous at some point? (6 pts.)
5. Show that if f and g are continuous, then so is their sum $f + g$. (8 pts.)
6. Show that if f and g are continuous, then so is their product $f \bullet g$. (10 pts.)
7. By applying the previous questions show that the function defined by

$$f(x) = x^2 - 4x + \sqrt{2}$$

is continuous. (5 pts.)
8. By using directly the definition of continuity show that the function defined by $f(x) = x^2 - 4x + \sqrt{2}$ is continuous. (10 pts.)
9. Let f and g be two functions. Assume that f is continuous at a and that g is continuous at $f(a)$. Show that $g \circ f$ is continuous at a. (15 pts.)
10. Let f be continuous at a and assume that $f(a) > 0$. Show that there is an interval around a where f is strictly positive. (15 pts.)
11. Let f be continuous. Assume that $f(a) < 0$ and $f(b) > 0$. Show that $f(c) = 0$ for some c between a and b. (17 pts.)