A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called **continuous at a point** \( a \in \mathbb{R} \), if for all \( \varepsilon > 0 \) (real or rational, it does not matter) there is a \( \delta > 0 \) such that for all \( x \in \mathbb{R} \), if \( |x - a| < \delta \) then \( |f(x) - f(a)| < \varepsilon \).

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called **continuous** if it is continuous at every point \( a \in \mathbb{R} \).

1. Show that a constant function is continuous. (3 pts.)
2. Show that the identity function is continuous. (5 pts.)
3. Is the function \( f \) defined by
   \[
   f(x) = \begin{cases} 
   -1 & \text{if } x < 0 \\
   1 & \text{if } x \geq 0 
   \end{cases}
   
   continuous? Justify your answer. (6 pts.)
4. Let \( f \) be defined as follows:
   \[
   f(x) = \begin{cases} 
   0 & \text{if } x \text{ is rational} \\
   1 & \text{otherwise} 
   \end{cases}
   
   Is \( f \) continuous at some point? (6 pts.)
5. Show that if \( f \) and \( g \) are continuous, then so is their sum \( f + g \). (8 pts.)
6. Show that if \( f \) and \( g \) are continuous, then so is their product \( f \cdot g \). (10 pts.)

7. By applying the previous questions show that the function defined by
   \[
   f(x) = x^2 - 4x + \sqrt{2}
   
   is continuous. (5 pts.)
8. By using directly the definition of continuity show that the function defined by
   \[
   f(x) = x^2 - 4x + \sqrt{2}
   
   is continuous. (10 pts.)
9. Let \( f \) and \( g \) be two functions. Assume that \( f \) is continuous at \( a \) and that \( g \) is continuous at \( f(a) \). Show that \( g \circ f \) is continuous at \( a \). (15 pts.)
10. Let \( f \) be continuous at \( a \) and assume that \( f(a) > 0 \). Show that there is an interval around \( a \) where \( f \) is strictly positive. (15 pts.)
11. Let \( f \) be continuous. Assume that \( f(a) < 0 \) and \( f(b) > 0 \). Show that \( f(c) = 0 \) for some \( c \) between \( a \) and \( b \). (17 pts.)