IBU
Math 121
Final Exam

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1. (12 points). Differentiate the following functions: arctan x, $\frac{\sin x}{e^{x^2}}$, $\sin^3(x^2)$

2. (8 points). Express sin(arctan x) and tan(arcsin x) as algebraic functions.

3. (12 points). Integrate

$\int_0^\pi x\cos x dx$, $\int \frac{\sin(\ln x)}{x} dx$, $\int \cos(\ln x)\,\mathrm{d}x$ (Hint: For the last one, apply integration by parts twice).

4. (15 points). Graph the function $f(x) = x/e^x$ with care.

5. (5 points). Let f be real-valued a function defined on an interval containing 0. Assuming that $1 \le f'(x) \le 2$ and $f(0) = 0$, show that $x \le f(x) \le 2x$ for all x in the domain of definition.

6. Let $x > 0$ and $a_n = x^n/n!$

6a. (4 points). Show that the sequence $(a_n)_n$ is decreasing after a while.

6b. (8 points). Show that $\lim_{n\to\infty} a_n = 0$. (Hint: Find an algebraic relation between a_n and a_{n+1}).

6c. (4 points). Conclude that $\lim_{n\to\infty} x^n/(n-1)! = 0$

7a. (8 points). Let $a_n \ge 0$. Assume that for some $r \in (0,1)$, $a_{n+1} < r\, a_n$ for large enough n. Deduce that the series $\sum_{i=0}^{\infty} a_i$ is convergent[1]. (Hint: Show that the partial sums $\sum_{i=0}^{n} a_i$ form an increasing and bounded sequence after a while).

7b. (8 points). Let a_n be real numbers. Show that if the series $\sum_{i=0}^{\infty} |a_i|$ is convergent, then so is $\sum_{i=0}^{\infty} a_i$. (Hint: Show that the partial sums form a Cauchy sequence).

7c. (8 points). Conclude from parts a and b that the series $\sum_{n=0}^{\infty} x^n / n!$ is convergent for all real numbers x.

7d. (8 points). Let $\exp(x) = \sum_{n=0}^{\infty} x^n / n!$ Show that

$$\exp(x)\exp(y) = \exp(x + y).$$

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[1] Recall that $\sum_{i=0}^{\infty} a_i$ means $\lim_{n\to\infty} \sum_{i=0}^{n} a_i$.