1. (12 points). Differentiate the following functions: \( \arctan x \), \( \frac{\sin x}{e^{x^2}} \), \( \sin^3(x^2) \).

2. (8 points). Express \( \sin(\arctan x) \) and \( \tan(\arcsin x) \) as algebraic functions.

3. (12 points). Integrate
\[
\int_0^\pi x \cos x \, dx, \quad \int \frac{\sin(ln x)}{x} \, dx, \quad \int \cos(ln x) \, dx
\]
(Hint: For the last one, apply integration by parts twice).

4. (15 points). Graph the function \( f(x) = \frac{x}{e^x} \) with care.

5. (5 points). Let \( f \) be real-valued a function defined on an interval containing 0. Assuming that \( 1 \leq f'(x) \leq 2 \) and \( f(0) = 0 \), show that \( x \leq f(x) \leq 2x \) for all \( x \) in the domain of definition.

6. Let \( x > 0 \) and \( a_n = \frac{x^n}{n!} \).

   6a. (4 points). Show that the sequence \( (a_n)_n \) is decreasing after a while.

   6b. (8 points). Show that \( \lim_{n \to \infty} a_n = 0 \). (Hint: Find an algebraic relation between \( a_n \) and \( a_{n+1} \)).

   6c. (4 points). Conclude that \( \lim_{n \to \infty} x^n/(n−1)! = 0 \).

7a. (8 points). Let \( a_n \geq 0 \). Assume that for some \( r \in (0,1) \), \( a_{n+1} < r a_n \) for large enough \( n \). Deduce that the series \( \sum_{i=0}^{\infty} a_i \) is convergent\(^1\). (Hint: Show that the partial sums \( \sum_{i=0}^{n} a_i \) form an increasing and bounded sequence after a while).

7b. (8 points). Let \( a_n \) be real numbers. Show that if the series \( \sum_{i=0}^{\infty} |a_i| \) is convergent, then so is \( \sum_{i=0}^{\infty} a_i \). (Hint: Show that the partial sums form a Cauchy sequence).

7c. (8 points). Conclude from parts a and b that the series \( \sum_{n=0}^{\infty} x^n / n! \) is convergent for all real numbers \( x \).

7d. (8 points). Let \( \exp(x) = \sum_{n=0}^{\infty} x^n / n! \) Show that \( \exp(x) \exp(y) = \exp(x + y) \).

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\(^1\) Recall that \( \sum_{i=0}^{\infty} a_i \) means \( \lim_{n \to \infty} \sum_{i=0}^{n} a_i \).