Math 121 (Calculus)

Midterm 1
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Write carefully using English and using it correctly.

Attempt to answer all the questions. You may use the previous questions in your answers.

You have 2 hours and 30 minutes.

1. Show that if $q \in \mathbb{Q}$ is a square in $\mathbb{Q}$, then $2q$ is not a square in $\mathbf{Q}$.

2. Show that for all rational number $q > 0$, there is a rational number x for which $0 < x^2 < q$.

3. Let $a < b$ be two rational numbers.

3a. Show that if x and y are two nonnegative rational numbers whose sum is 1, then $a \leq ax + by \leq b$.

3b. Let q be a rational number such that $a \leq q \leq b$. Show that there are two nonnegative rational numbers x and y such that $x + y = 1$ and $q = ax + by$.

3c. Show that the numbers x and y of 3b are unique.

4. Recall that a **cut** in $\mathbb{Q}$ is a nonempty proper subset U of $\mathbb{Q}$ such that

a) if $x \in U$ and $y > x$ then $y \in U$.

b) U does not have a least element.

Let U be a cut in $\mathbb{Q}$.

4a. Show that the set $\{u^2 : u \in U\}$ is never a cut.

4b. Show that if $0 \in U$, then the set $\{q \in \mathbb{Q} : q \geq u^2 \text{ for some } u \in U\}$ is never a cut. (**Hint:** What is this set?)

4c. Show that the set $\{q \in \mathbb{Q} : q > u^2 \text{ for some } u \in U\}$ is a cut.

4d. Show that if $0 \notin U$, then,

$$\{q \in \mathbb{Q} : q > u^2 \text{ for some } u \in U\} = \{uv : u \in U \text{ and } v \in U\}.$$