MATH 212

Basic Algebra II

Ali Nesin

1) Find all ring homomorphisms from $\mathbb{R}$ into $\mathbb{R}$.

2) Let n be an natural number. Find all ring homomorphisms from $\mathbb{R}^n$ into $\mathbb{R}$.

3) Show that the additive group of a commutative ring with identity cannot be isomorphic to the additive group $\mathbb{Q}/\mathbb{Z}$.

4) Let R be a principal ideal domain. And let I and J be nonzero ideals in R. Show that $IJ = I \cap J$ if and only if $I + J = R$.

5) Show that for any prime $p > 2$, $(\mathbb{Z}/p\mathbb{Z})[X]$ has an irreducible polynomial of degree 2. Is the statement true for $p = 2$?

6) Show that the equation $x^3 = 1$ has three distinct roots in a field if and only if -3 has a square root and if characteristic of the field is not 3.

7) Is the ideal of $\mathbb{Z}[X]$ generated by $X^3 + X + 1$ prime? (An ideal I of a commutative ring R is called **prime** if R/I is a domain)

8) Let I be an ideal and S be a subring of the ring R. Prove that $I \cap S$ is an ideal of S. Give an example to show that every ideal of S need not be of the form $I \cap S$ for some ideal I of R.

9) Let R be a commutative ring and P be a maximal ideal of R. Let $I = P[X]$ be the ideal of the polynomial ring $R[X]$ consisting of the polynomials in $R[X]$ with coefficients in P. Show that I is prime ideal that is not a maximal ideal.

10) Let R be a commutative ring with identity. Let $f(X) = a_0 + a_1X + ... + a_n X^n \in R[X]$. Prove that $f(X)$ is unit if and only if a_0 is a unit in R and a_i is nilpotent for all $i > 0$.

11) Find all irreducible $\mathbb{Z}/60\mathbb{Z}$-modules.

12) Let R be a domain and K its field of fractions. Show that an R-submodule of K is indecomposable.

13) Find all ring automorphisms of $\mathbb{Q}[X]$.

14) Find all field automorphisms of $\mathbb{Q}(X)$.