Metamathematics of Elementary Mathematics

Lecture 7

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Previous case studies:

- **Adding** $7 + 5$
- **Dividing** 10 **apples between** 5 **people**

Adding $7 + 5$

EHK, aged 6:

- could easily solve "put a number in the box" problems of the type

  $$7 + \square = 12,$$

  by counting how many 1's she had to add to 7 in order to get 12

- but struggled with

  $$\square + 6 = 11,$$

  because she did not know where to start.

Dividing 10 apples between 5 people

The First Law of Arithmetic: you do not add fruit and people.

Giuseppe Arcimboldo, *Autumn*. 1573. Musée du Louvre, Paris.

If we cannot *add* apples and people, why can we *divide* apples by people:

$$10\,\text{apples} \,:\, 5\,\text{people} = 2\,\text{apples}.$$

When we distribute 10 apples giving 2 apples to a person, we have

$$10\,\text{apples} \,:\, 2\,\text{apples} = 5\,\text{people}$$

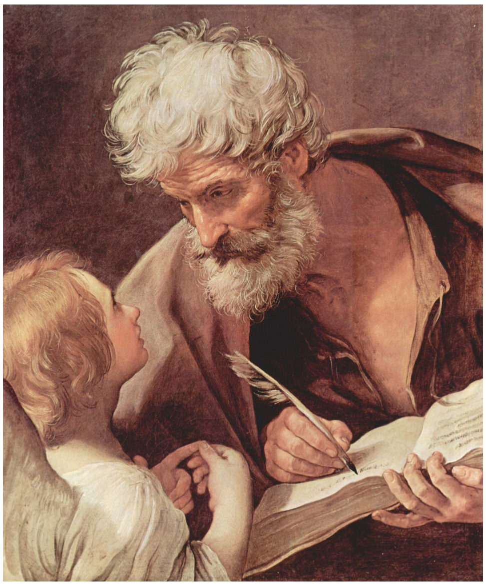
Where do "people" on the right hand side of the equation come from?

Why do "people" appear and not, say, "kids"?

There were no "people" on the left hand side of the operation!

How do numbers on the left hand side know the name of the number on the right hand side?

A new story: multiplying long numbers



L'Evangelista Matteo e l'Angelo. Guido Reni, 1630–1640.
Pinacoteca Vaticana.

Study of children reveals more about mathematical thinking than any number of observations of adults.

What's weird about 1, 11, 111, 1111 etc when you square them?

$1^2=1$. $11^2=121$. Keep on doing this with the other numbers. (If necessary use a calculator).

Solutions see page it counts up e.g. $1111111^2 = 1234567654321$

But when you have $1{,}111{,}111{,}111^2$ the answer is different. Figuring out (or using the calculator) what are the next square numbers in the pattern after 1111111111^2?

Solutions see page 1234567900987654321, 123456790120987654321, 12345679012320987654321, 1234567901234320987654321 and 123456790123454320987654321!

Do you notice a pattern?

From a workbook of a schoolboy DW, aged 8.

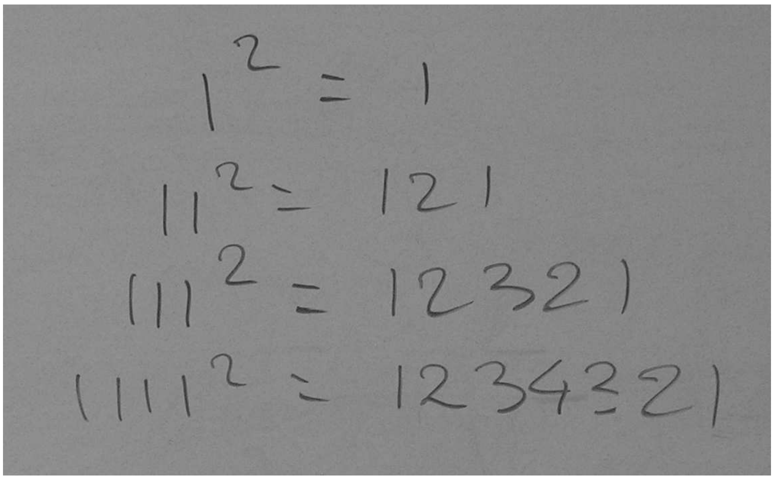
$$
\begin{aligned}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 123454321 \\
111111 \times 111111 &= 12345654321 \\
1111111 \times 1111111 &= 1234567654321 \\
11111111 \times 11111111 &= 123456787654321 \\
111111111 \times 111111111 &= 12345678987654321
\end{aligned}
$$

This is what DW meant.

But the pattern breaks at the next step and DW has already noticed that:

$$1111111111 \times 1111111111 = 1234567900987654321$$

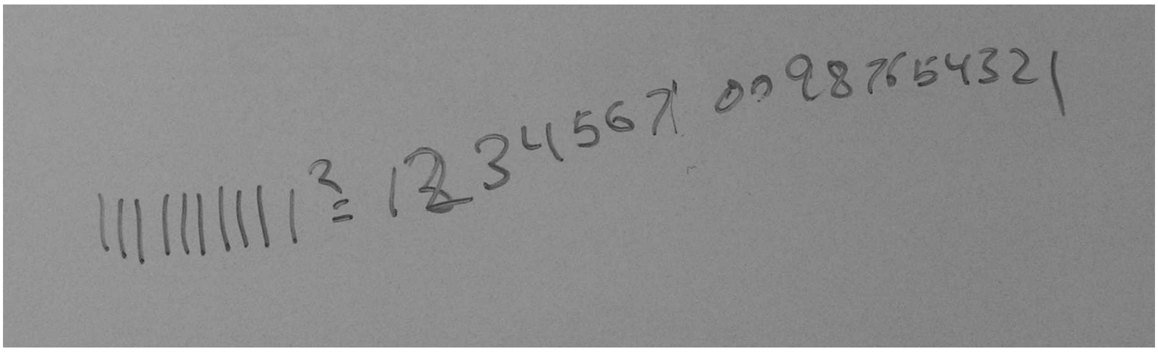
The result is no longer symmetric. Why?



My writing on a whiteboard during my first meeting with DW.

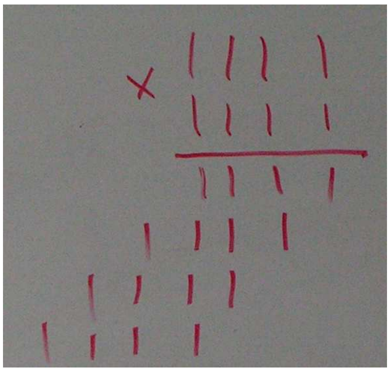
I asked DW whether the symmetric pattern of results continued indefinitely.

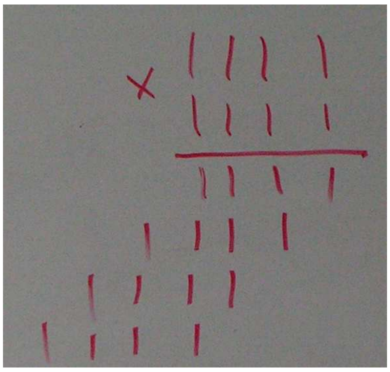
DW instantly answered “No”.



To illustrate his point, DW's wrote down, apparently from his memory, the first case when the pattern breaks.

“Good”–said I–but let us try to figure out why this is happening”, and wrote on the board:

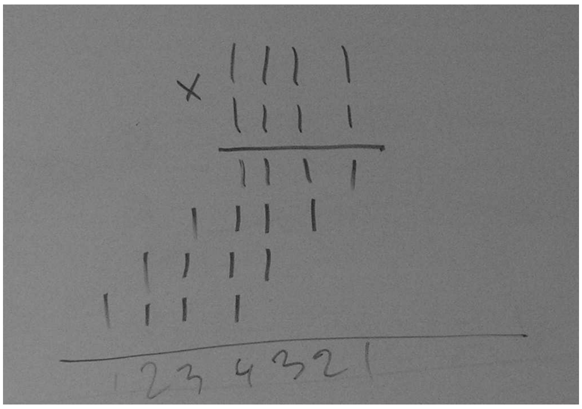




“Yes”–said DW–“this is column multiplication”.

“And what are the sums of columns’?”

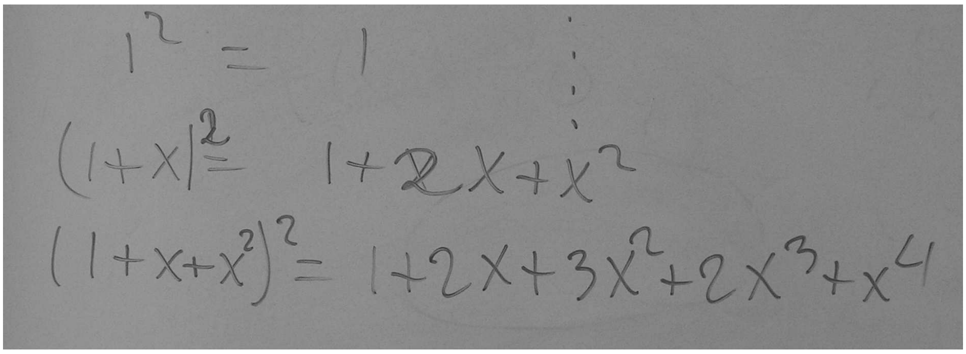
“1, 2, 3, 4, 3, 2, 1”–dictated DW to me, and I have written down the result:



me: "Will the symmetric pattern continue indefinitely?"

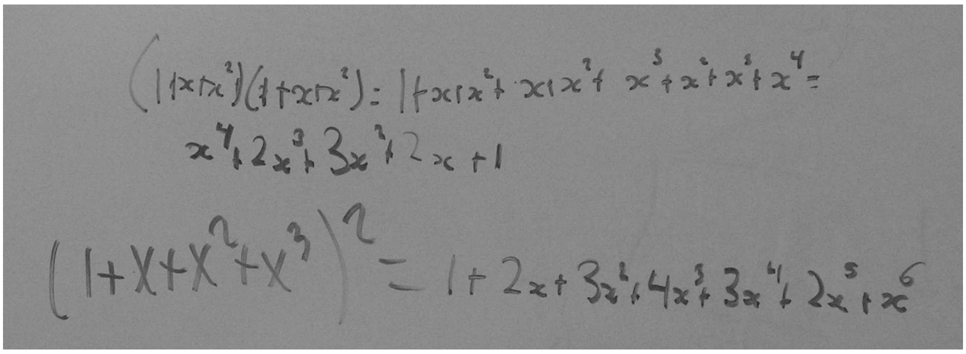
DW: "No – when there are 10 1's in a column, 1 is added on the left and there is no symmetry."

me: "Yes! Carries break the symmetry. But let us look at another example"–and I wrote:



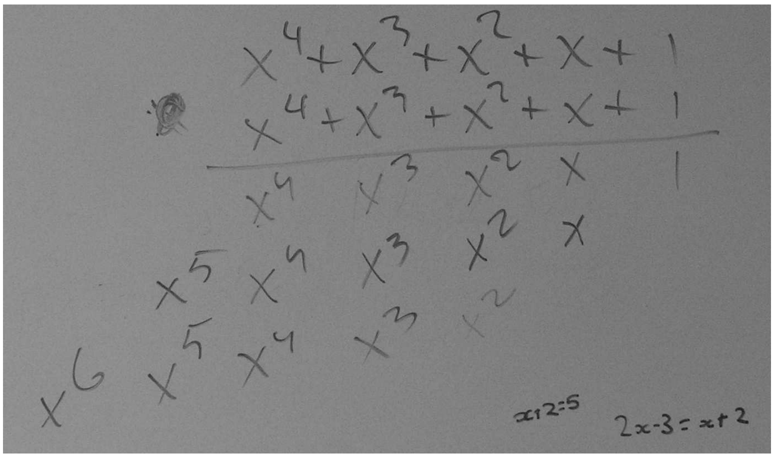
DW was intrigued and made a couple of experiments:

It appeared from his behaviour that he was using mostly mental arithmetic, writing down the result, term by term, with pauses:

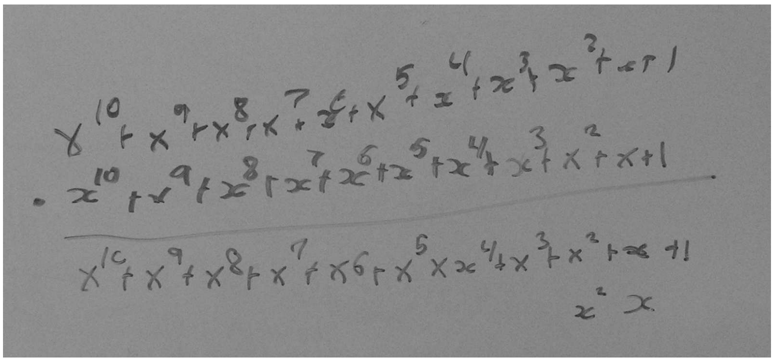


DW said with obvious enthusiasm: "Yes, it is the same pattern!"

“Wonderful”–answered I–“let us see why this is happening. I’ll give you a hint: multiplication of polynomials can be written as column multiplication”, and started to write:



DW did not let me finish, grabbed the marker from my hand and insisted on doing it himself:



He stopped after he barely started the second line and said very firmly: "Yes, it is like with numbers".

me: "Well – but will the pattern break down or will continue forever?"

[Pause. . .]

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[Pause. . .]

DW: "No, it will not break down!"

me: "Why?"

DW: "Because when you add polynomials, the coefficients just add up, there are no carries."

me: "You know, in mathematics polynomials are sometimes used to explain what is happening with numbers".

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DW: “Yes, 10 is x.”

Even elementary arithmetic conceals sophisticated hidden structures

Indeed, a carry, a digit that is transferred from one column of digits to another column of more significant digits during addition of two decimals, is defined by the rule

$$c(a,b) = \begin{cases} 1 & \text{if } a+b>9 \\ 0 & \text{otherwise} \end{cases} .$$

Carry is a 2-cocycle from $\mathbb{Z}/10\mathbb{Z}$ to $10\mathbb{Z}$

$$c : \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \longrightarrow 10\mathbb{Z}$$

and is responsible for the extension of additive groups

$$0 \longrightarrow 10\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0.$$

DW does not know cohomology (yet), but he is alert to potential presence of sophisticated hidden structures.

Let G be a group and A an abelian group.

A 2-**cocycle** is a map

$$f : C \times G \longrightarrow A$$

such that

$$g \cdot f(h, k) + f(g, hk) = f(gh, k) + f(g, h)$$

Let E be an extension of A by G,

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1,$$

S a system of coset representative of A in E,

$$s : G \longrightarrow E$$

a coset map.

Then $f : G \times G \to A$ defined by

$$s(gh) = f(g,h)s(g)s(h)$$

is a 2-cocycle.

It measures the extent to which the collection of coset representatives fails to be closed under multiplication.

A 2-coboundary for the action of G on A, is a function

$$f : G \times G \to A$$

such that there exists a function

$$\phi : G \to A$$

such that:

$$f = (g, h) \mapsto g \cdot \phi(h) - \phi(gh) + \phi(g)$$

Two 2-cocycles for the same extension E differ by a 2-coboundary.

$$H^2(G, A) = Z^2(G, A)/B^2(G, A)$$

10-**adic numbers:**

- decimals infinite to the left:

$$\cdots 555554444333221,$$

  with usual addition and multiplication.

- Projective limit of

$$\cdots \to \mathbb{Z}/10^3\mathbb{Z} \to \mathbb{Z}/10^2\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} \to 0$$

Exercise: Find zero divisors, explicitly.

For p prime, p-adics are better.

Arithmetical carry contains in itself a germ of a *consistent and rigorous* concept of limit.

Euler:

$$1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots = -1.$$

Does it make sense? Did it make sense for Euler?

Gottfried Leibniz: fully documented modern binary number system in the 17th century in *Explication de l'Arithétique Binaire*.

Leibniz's system used 0 and 1, like the modern binary numeral system.

$$\begin{aligned}
0 &= 0\\
1 &= 1\\
2 &= 10\\
3 &= 11\\
4 &= 100\\
5 &= 101\\
6 &= 110\\
7 &= 111\\
&\vdots\\
2^n &= 10\cdots0 \qquad (\ n \text{ zeroes})
\end{aligned}$$

TABLE DES NOMBRES.

0 0 0 0 0 0	0
0 0 0 0 0 1	1
0 0 0 0 1 0	2
0 0 0 0 1 1	3
0 0 0 1 0 0	4
0 0 0 1 0 1	5
0 0 0 1 1 0	6
0 0 0 1 1 1	7
0 0 1 0 0 0	8
0 0 1 0 0 1	9
0 0 1 0 1 0	10
0 0 1 0 1 1	11
0 0 1 1 0 0	12
0 0 1 1 0 1	13
0 0 1 1 1 0	14
0 0 1 1 1 1	15
0 1 0 0 0 0	16
0 1 0 0 0 1	17
0 1 0 0 1 0	18
0 1 0 0 1 1	19
0 1 0 1 0 0	20
0 1 0 1 0 1	21
0 1 0 1 1 0	22
0 1 0 1 1 1	23
0 1 1 0 0 0	24
0 1 1 0 0 1	25
0 1 1 0 1 0	26
0 1 1 0 1 1	27
0 1 1 1 0 0	28
0 1 1 1 0 1	29
0 1 1 1 1 0	30
0 1 1 1 1 1	31
1 0 0 0 0 0	32
&c.	&c.

bres entiers au-dessous du double du plus haut degré. Car icy, c'est comme si on disoit, par exemple, que 111 ou 7 est la somme de quatre, de deux & d'un.

100	4
10	2
1	1
111	7

Et que 1101 ou 13 est la somme de huit, quatre & un. Cette proprieté sert aux Essayeurs pour peser toutes sortes de masses avec peu de poids, & pourroit servir dans les monnoyes pour donner plusieurs valeurs avec peu de pieces.

1000	8
100	
1	1
1101	13

Cette expression des Nombres étant établie, sert à faire tres-facilement toutes sortes d'operations.

Pour l'*Addition* par exemple. ☽

110	6	101	5	1110	14
111	7	1011	11	10001	17
1101	13	10000	16	11111	31

Pour la *Soustraction.*

1101	13	10000	16	11111	31
111	7	1011	11	10001	17
110	6	101	5	1110	14

Pour la *Multiplication.* ⊙

11	3	101	5	101	5
11	3	11	3	101	5
11		101		101	
11		101		1010	
1001	9	1111	15	11001	25

Pour la *Division.*

15 ‖ 1 1 1 1 ⎫ 1 0 1 ‖ 5
3 ‖ 1 1 1 ⎬
1 1 ⎭

Et toutes ces operations sont si aisées, qu'on n'a jamais besoin de rien essayer ni deviner, comme il faut faire dans la division ordinaire. On n'a point besoin non-plus de rien apprendre par cœur icy, comme il faut faire dans le calcul ordinaire, où il faut sçavoir, par exemple, que 6 & 7 pris ensemble font 13; & que 5 multiplié par 3 donne 15, suivant la Table *d'une fois un est un*, qu'on appelle Pythagorique. Mais icy tout cela se trouve & se prouve de source, comme l'on voit dans les exemples precedens sous les signes ☽ & ⊙.

Back to Euler:

$$\begin{aligned}1+2+4+8+\cdots &= 1+10+100+1000+\cdots \\ &= \cdots 111111111\end{aligned}$$

$$\begin{array}{rr} & \cdots 111111111 \\ + & 1 \\ = & \cdots 000000000 \end{array}$$

Hence in 2-adic arithmetic

$$1+2+4+8+\cdots = -1$$