Metamathematics of Elementary Mathematics
Lecture 5
Navigation on the Riemann Sphere

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based on ideas of Hovik Khudaverdyan

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Şirince,
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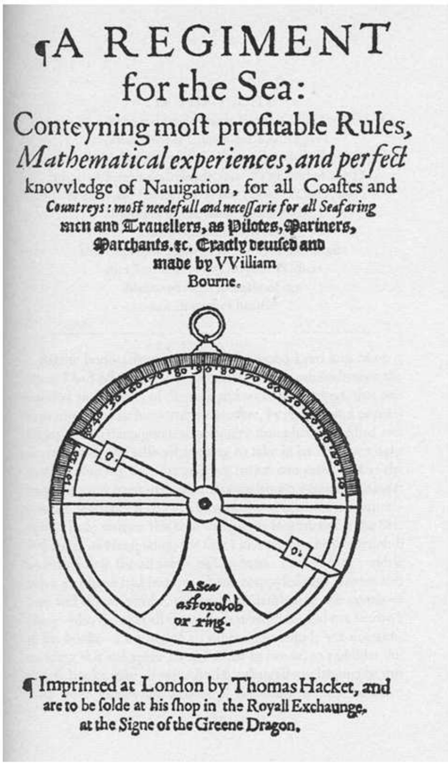
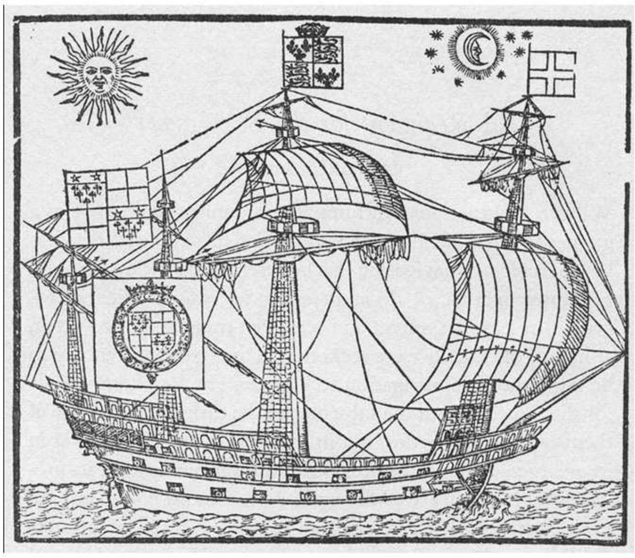


¶A REGIMENT
for the Sea:
Conteyning most profitable Rules,
Mathematical experiences, and perfect
knowledge of Nauigation, for all Coastes and
Countreys: most needefull and necessarie for all Seafaring
men and Trauellers, as Pilotes, Mariners,
Marchants, &c. Exactly deuised and
made by VVilliam
Bourne.

¶ Imprinted at London by Thomas Hacket, and
are to be solde at his shop in the Royall Exchaunge,
at the Signe of the Greene Dragon.

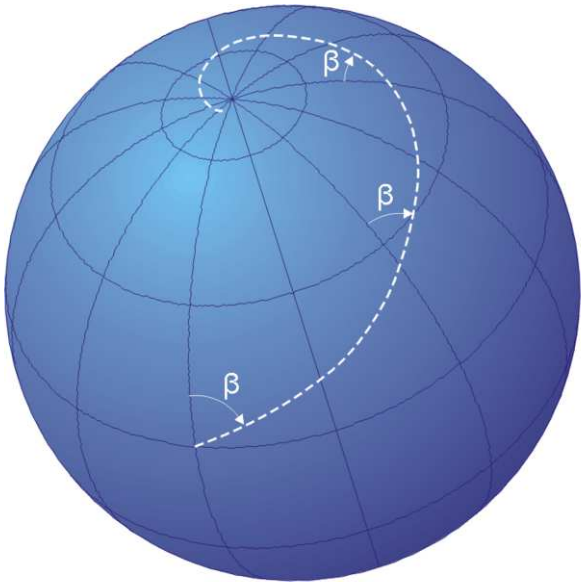


The first & most principal thing for any seafaring man or traveller, is to know toward what part of the Earth he meaneth to go.

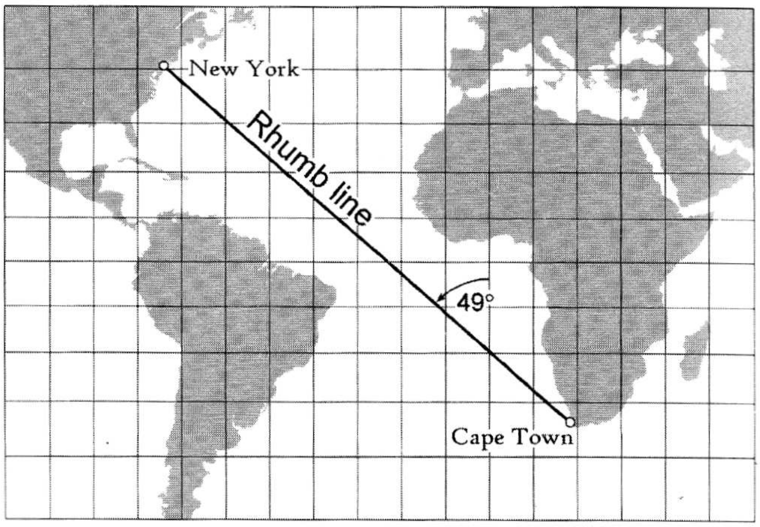
A heroic era of seafaring with astrolabe and compass.

A ship compass in a Cardano suspension. A modern replica.

An astrolabe.

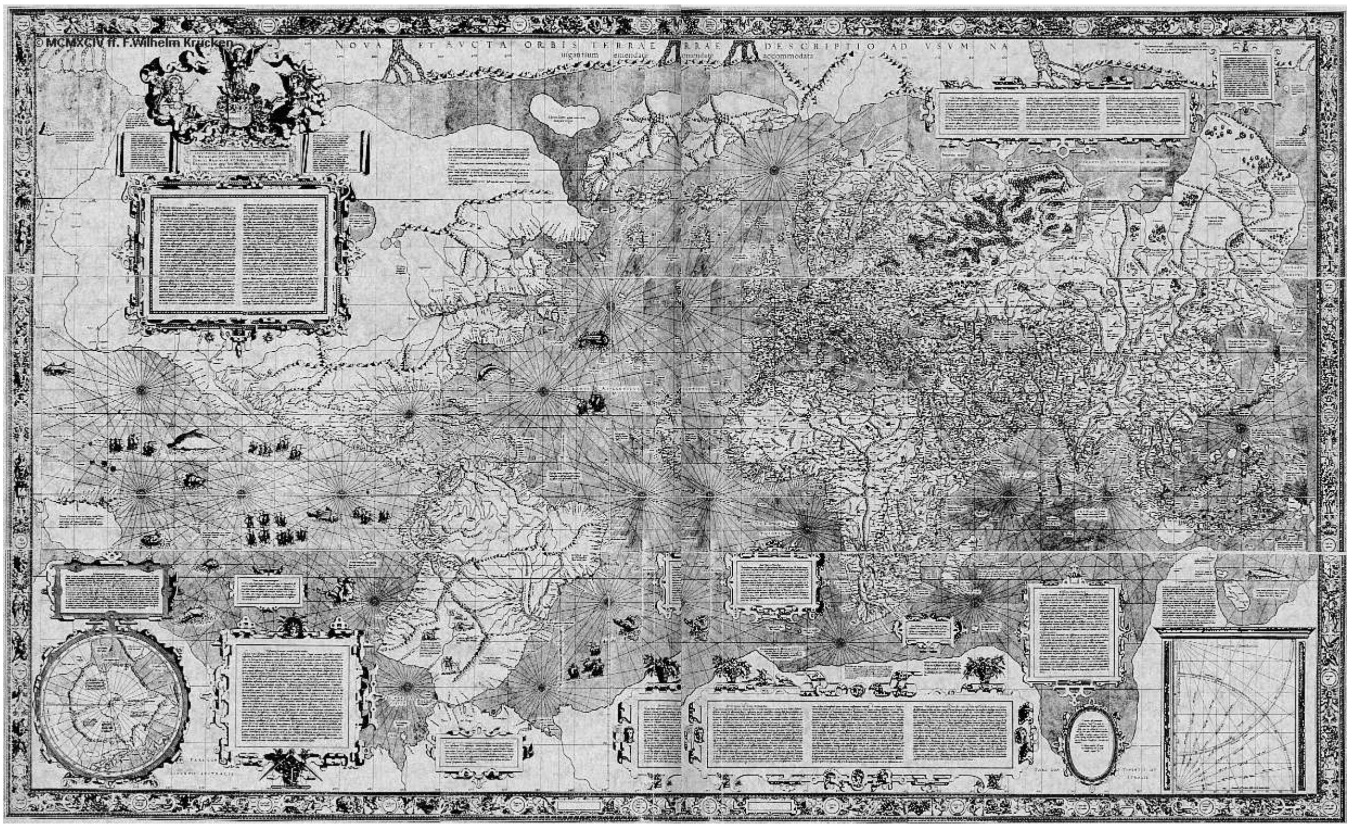


Loxodrome: a path of constant direction on the globe.

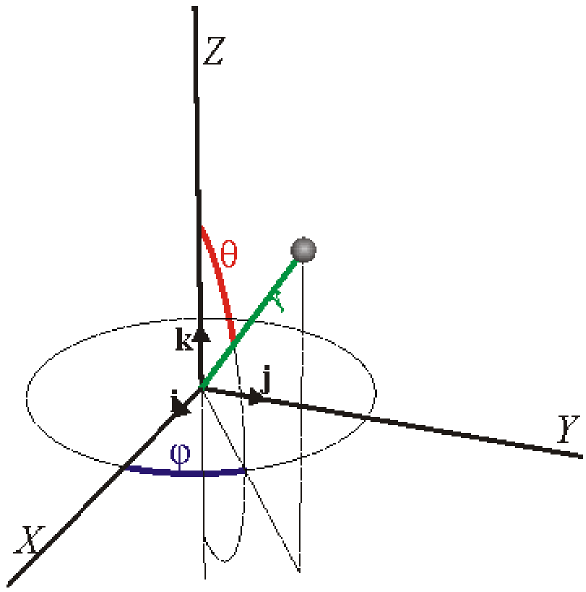


Rhumb lines: lines of constant direction on a map.
How to make them straight?

Gerard Mercator (1512–1594).



Mercator's famous map, 1569: rhumb lines are straight.



Spherical coordinates on the sphere $x^2 + y^2 + z^2 = R^2$:

$$x = R\sin\theta\cos\varphi, \quad y = R\sin\theta\sin\varphi, \quad z = R\cos\theta$$

$(\theta(t), \varphi(t))$ – a curve on the sphere

tangent vector (x_t, y_t, z_t) at point (θ, φ):

$R(\cos\theta \cdot \cos\varphi \cdot \theta_t - \sin\theta \cdot \sin\varphi \cdot \varphi_t, \cos\theta \cdot \sin\varphi \cdot \theta_t + \sin\theta \cdot \cos\varphi \cdot \varphi_t, -\sin\theta \cdot \theta_t),$

Meridian: $\theta = t, \varphi = \mathrm{const}$

tangent vector to meridian:

$$R(\cos\theta \cdot \cos\varphi, \cos\theta \sin\varphi, -\sin\theta).$$

Cosine of angle $\alpha(t)$ between the curve and the meridian:

$$\cos\alpha(t) = \frac{\theta_t(t)}{\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2(t)}} \tag{1}$$

Loxodrome:

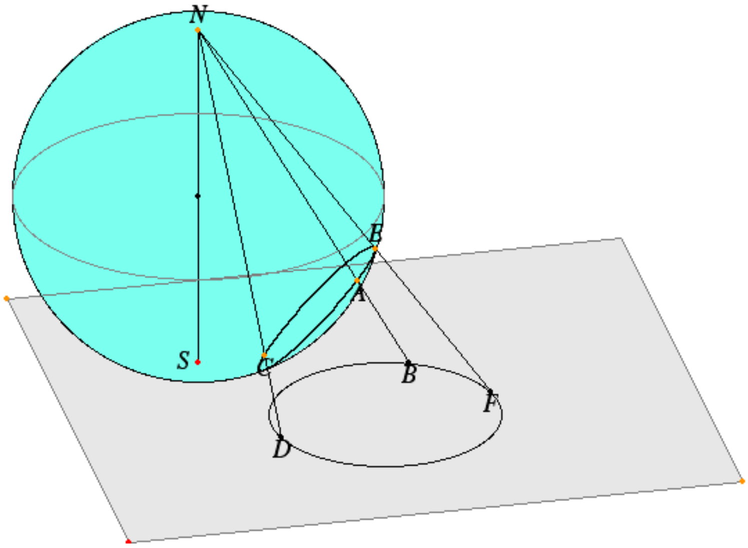
$$\frac{\theta_t(t)}{\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2(t)}} = \text{const}$$

$$\frac{\theta_t}{\varphi_t} = \pm\frac{c}{1-c^2}\sin\theta$$

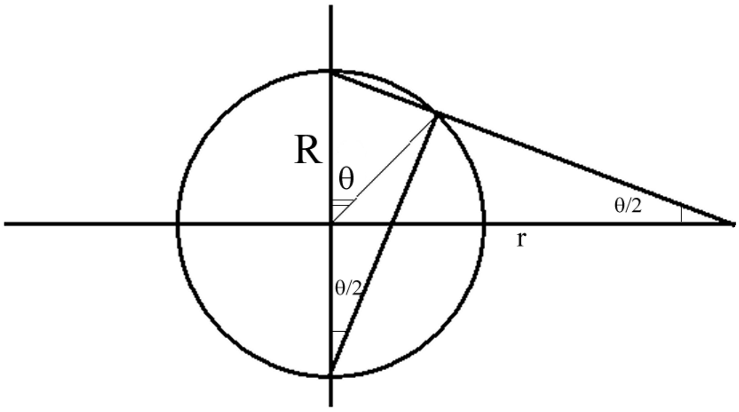
Change of variables: $t = t(\varphi)$, $\theta = \theta(\varphi)$

$$\frac{d\varphi}{d\theta} = \pm\frac{1-c^2}{c}\cdot\frac{1}{\sin\theta}$$

$$\varphi(\theta) = k\int\frac{d\theta}{\sin\theta} = k\log\tan\frac{\theta}{2} + \varphi_0$$



Stereographic projection from the North Pole onto the plane tangent at the South Pole. Invented or known to Apollonius of Perga?



$$\frac{R}{r} = \tan\frac{\theta}{2}$$

Choose polar coordinates r, φ in the plane, with φ being the same as in spherical coordinates. Then stereographic projection sends

$$(\theta, \varphi) \mapsto (\frac{R}{\tan\frac{\theta}{2}}, \varphi)$$

Since

$$\tan\frac{\theta}{2} = \frac{R}{r},$$

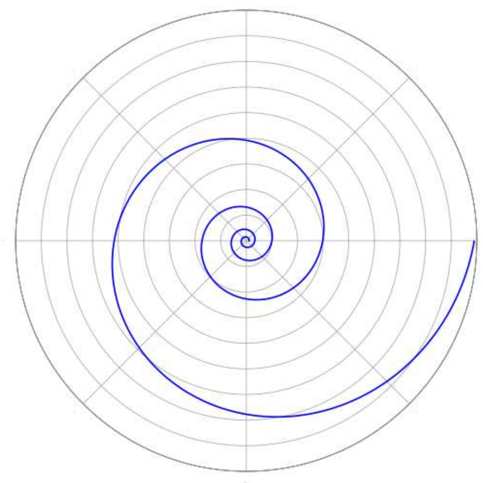
the image of loxodrome

$$\varphi(\theta) = k \log\tan\frac{\theta}{2} + \varphi_0$$

becomes

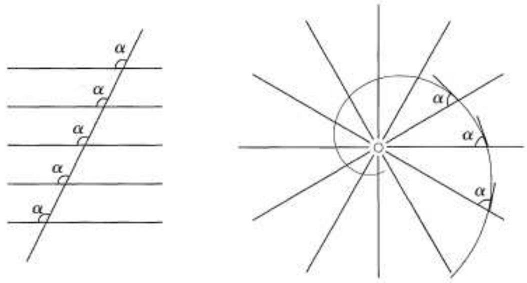
$$\varphi = -k \log\frac{r}{R} + \varphi_0.$$

This is logarithmic spiral.



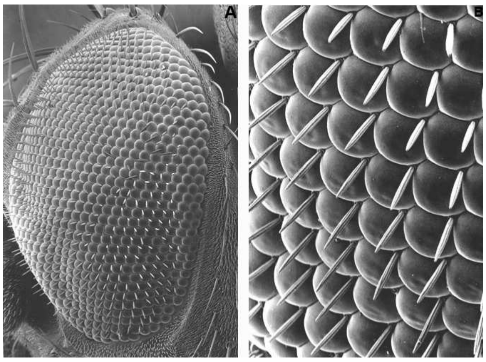
Logarithmic spiral

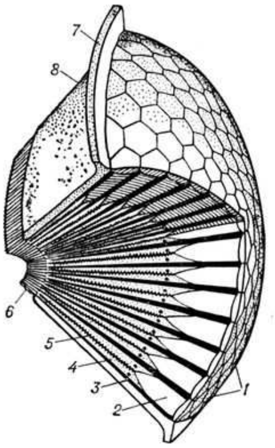
$$\varphi = -k \log \frac{r}{R} + \varphi_0.$$



Logarithmic spiral is characterised by the property that it intersects radial lines at constant angles.

Insects fly to a candle along a logarithmic spiral.





Ommatidia in a facet eye of an insect control bearing at source of light.

A spider's web. The same principle of constant angles?

Move to complex variables

$$Z = u + iv = re^{i\varphi}$$

Then the map

$$W = s + it = \log \frac{Z}{R}$$

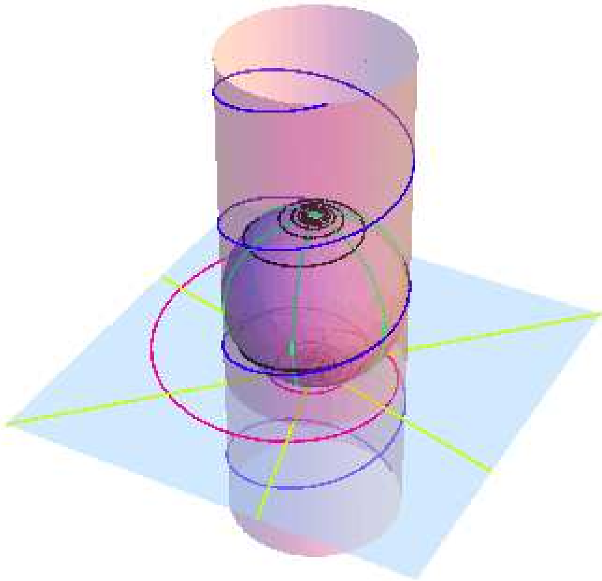
$$W = \log \frac{Z}{R} = \log \frac{r}{R} + i\varphi,$$

sends logarithmic spirals

$$\varphi = k \log \frac{r}{R} + \varphi_0.$$

to straight lines $t = ks + t_0$.

Meridians $\varphi = \varphi_0$ are sent to straight lines $t = \varphi_0$ and parallels $\theta = \theta_0$ to segments $s = \log \tan \frac{\theta_0}{2}$.



A loxodrome, a logarithmic spiral and a rhumb line on a cylinder. (Dr G Megyesi).

Why cylinder?

Bernhard Riemann (1826–1866).

Riemann sphere: identified with the complex plane by stereographic projection.

Leonard Euler (1707–1783).

$$z \mapsto e^z,$$

is periodic,

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot 1 = e^z,$$

because

$$e^{\pi i} = -1,$$

Now Mercator's projection is nothing else but the logarithm itself:

$$Z \mapsto \operatorname{Log} Z.$$

Conformal map: a map from a surface to surface which preserves angles.

Map

$$\begin{aligned} \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto az \end{aligned}$$

is conformal.

If $w = F(z)$ from $\mathbb{C}$ to $\mathbb{C}$ is differentiable then

$$\Delta w \approx F'(z)\Delta z,$$

and F is conformal.

An orientation-preserving conformal map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is an analytic function.

Indeed, the Jacobian

$$(x, y) \mapsto (u, v)$$

i spositive. The Jacobi matrix:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

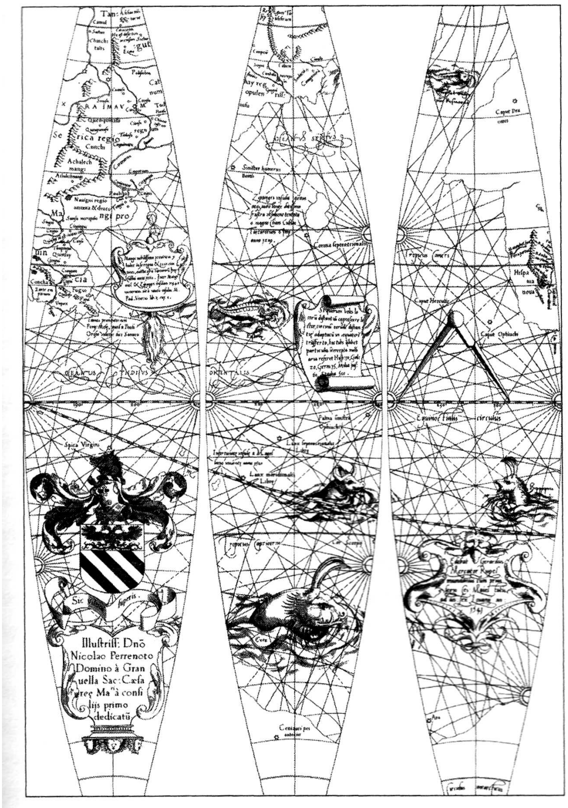
is aproduct of scalar and orthogonal matrices. This is exactly Cauchy-Riemann condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The rest is the classical theory of the functions of complex variable.

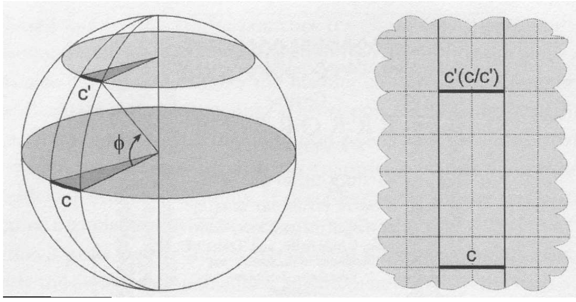
How all that was done?

Mercator's globe, 1541.

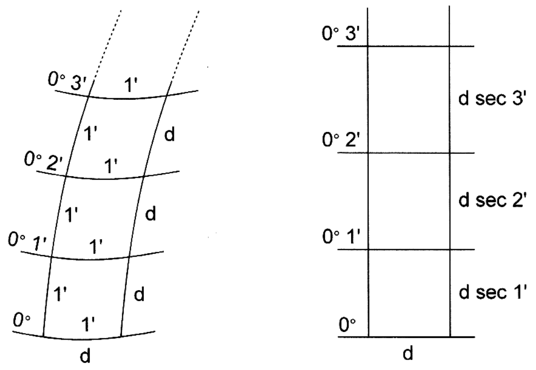


Segments of Mercator's globe

Edward Right, *The Correction of Certain Errors in Navigation*, 1599.



$$\frac{c}{c'} = \frac{1}{\cos\phi} = \sec\phi.$$



Accumulation of corrections:

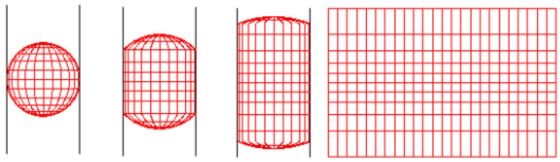
$$d \sec 1', \quad , d \sec 2', \quad , d \sec 3', \ldots$$

$$Y(\phi) = d(\sec 1' + \sec 2' + \sec 3' + \cdots + \sec \phi).$$

Wright just summed up the table of secants.

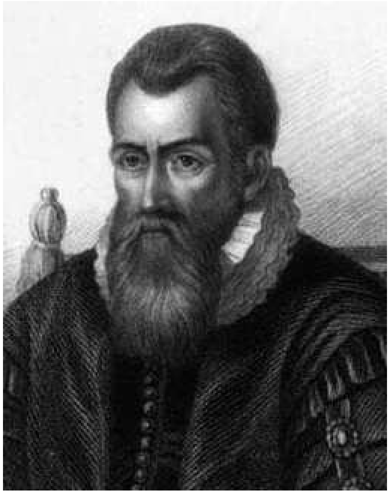
In modern terminology:

$$Y(\phi) = \int_0^\phi \sec t dt.$$



Suppose a sphericall superficies with meridians, paralels, rumbs, and the whole hydrographicall description drawne thereupon to bee inscribed into a concave cylinder, their axes agreeing in one. Let this sphericall superficies swel like a bladder, (whiles it is in blowing) equally alwayes in euerie part thereof (that is as much in longitude as in latitude) till it apply, and ioyne it selfe (round about, and all alongst also towardes either pole) vnto the concave superficies of the cylinder : each paralel vpon this sphericall superficies increasing successively from the equinoctiall towardes eyther pole, vntil it come to bee of equall diameter with the cylinder, and consequently the meridians stil widening them selves, til they come to be so far distant euery where ech from other as they are at the Equinoctiall.

Edward Wright was thinking in terms of conformal maps!



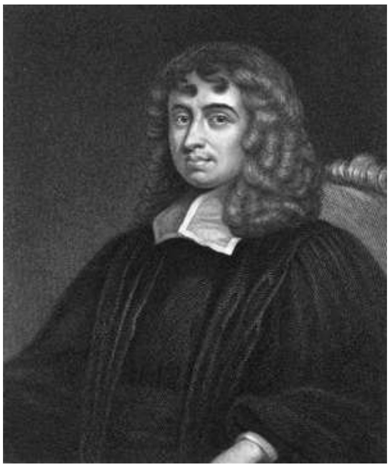
John Napier (1550–1617)

Henri Bond, 1644: noticed, by looking into tables of logarithm of tangents, that

$$Y(\phi) = \log(\tan(\phi/2 + \pi/4))$$

(which is the same as $\log \tan \frac{\theta}{2}$)

For years remained an open problem.



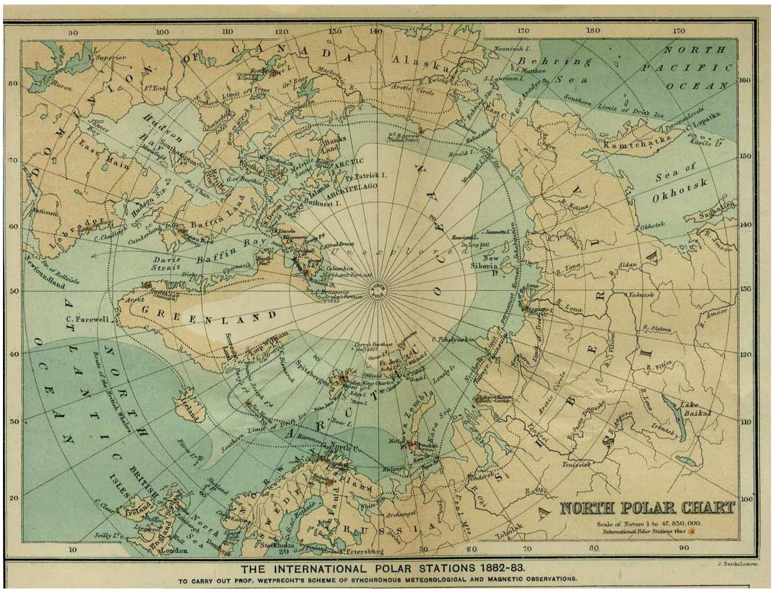
Isaac Barrow (1630–1677)

Finally proved

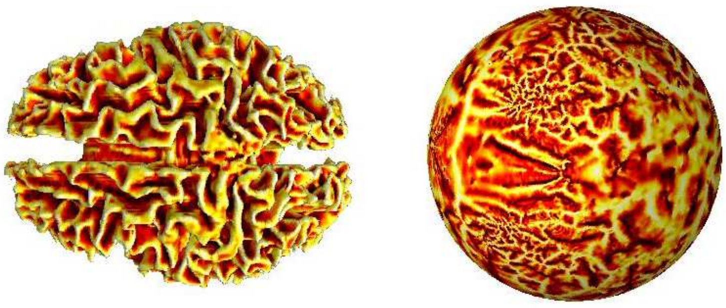
$$\int_0^{\phi} \sec(t)\, dt = \log\left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)$$

Is Mercator's projection misleading?

New look at the changing Earth



Map of the North Pole, 1885.
Notice a huge blank space.



Modern developments: conformal map of the surface of brain on a sphere.

From cartography to colonoscopy

Virtual colonoscopy has some fundamental problems, which it shares with conventional colonoscopy. The most important one is that the **navigation** using inner views is very challenging . . .

We present a method for **mapping** the colon onto a flat surface [. . .] based on a certain mathematical technique from **Riemann surface** theory, which allows us to map any highly undulating tubular surface without handles or self-intersections onto a planar rectangle in a **conformal manner**.

[S. Haker et al. *Non-distorting Flattening for Virtual Colonoscopy*, Lect. Notes Comp. Sci. 1935 (2000) 358–366.]

Thank you very much for your attention!