## Part 3. Carry, Cinderella of Arithmetic

### 10. Palindromic decimals and palindromic polynomials

My next case study is based on conversations with an 8 year old boy, DW, in May 2007.

DW's parents sent me a file of DW's book. It included the following paragraph, reproduced here *verbatim*:

> **What's weird about 1, 11, 111, 1111 etc when you square them?**
>
> $1^2$=1. $11^2$=121. Keep on doing this with the other numbers. (If necessary use a calculator).
>
> Solutions see page it counts up e.g. $1111111^2$ = 1234567654321
>
> But when you have $1{,}111{,}111{,}111^2$ the answer is different. Figuring out (or using the calculator) what are the next square numbers in the pattern after $1111111111^2$?
>
> Solutions see page 1234567900987654321, 123456790120987654321, 12345679012320987654321, 1234567901234320987654321 and 123456790123454320987654321!
>
> Do you notice a pattern?

I wrote to DW:

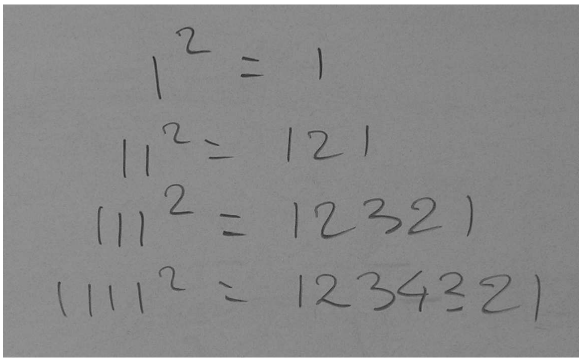
> Dear D,
>
> Indeed, there is something weird. I believe you have figured out that
>
> $$\begin{array}{rcl} 1\times 1 &=& 1\\ 11\times 11 &=& 121\\ 111\times 111 &=& 12321\\ 1111\times 1111 &=& 1234321\\ 11111\times 11111 &=& 123454321\\ 111111\times 111111 &=& 12345654321\\ 1111111\times 1111111 &=& 1234567654321\\ 11111111\times 11111111 &=& 123456787654321\\ 111111111\times 111111111 &=& 12345678987654321 \end{array}$$
>
> There is a wonderful palindromic pattern in the results. But mathematics is interested not so much in beautiful patterns but in reasons why the patterns cannot be extended without loss of their beauty. In our case, the pattern breaks at the next step (judging by your book, you have already noticed that):
>
> $$1111111111 \times 1111111111 = 1234567900987654321$$

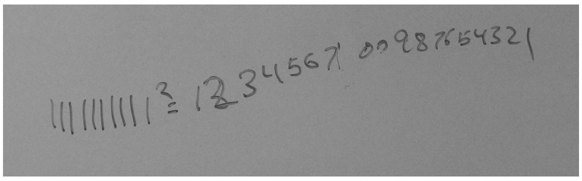
> The result is no longer symmetric. Why? What is the difference from the previous 9 squares? Can you give any suggestions?

I had some brief e-mail exchanges with DW which suggested that he might have an explanation, but could not clearly express himself. Our discussion continued when he visited me (with his mother) in Manchester on 8 May 2007.

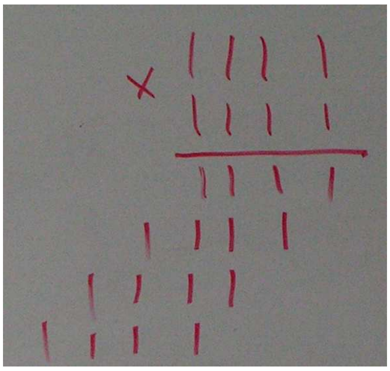
I wrote on the whiteboard in my office (this is a photograph of actual writing on the board):




and asked DW whether the symmetric pattern of results continued indefinitely. DW instantly answered "No" and also instantly wrote on the board, apparently from his memory:



"Good"–said I–but let us try to figure out why this is happening", and wrote on the board:

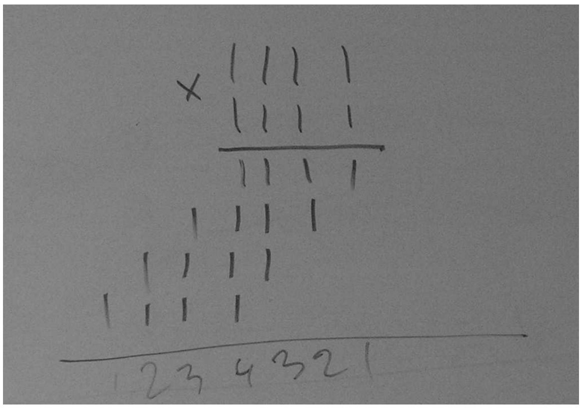


"Yes"–said DW–"this is column multiplication".

"And what are the sums of columns'?"

"1, 2, 3, 4, 3, 2, 1"–dictated DW to me, and I have written down the result:

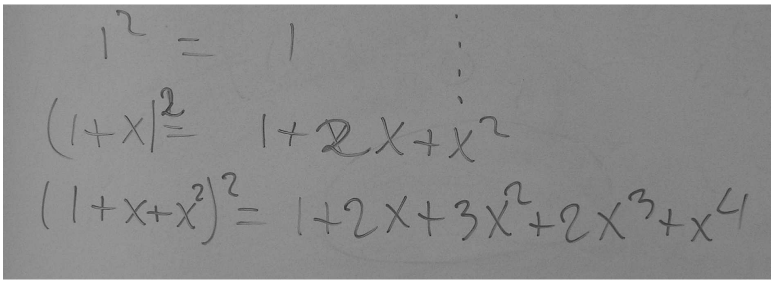
 



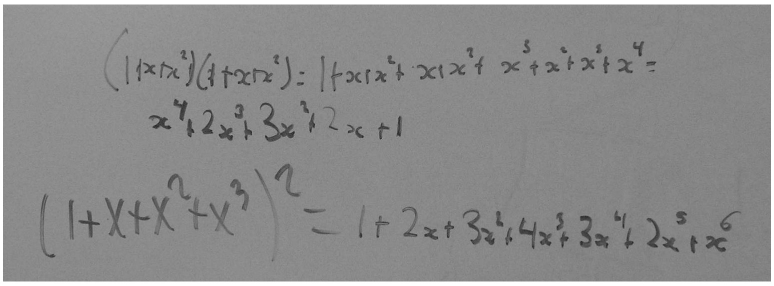

"Will the symmetric pattern continue indefinitely?"–asked I.

"No"–was DW's answer–"when there are 10 1's in a column, 1 is added on the left and there is no symmetry."

"Yes!"–said I–"carries break the symmetry. But let us look at another example"–and I wrote:



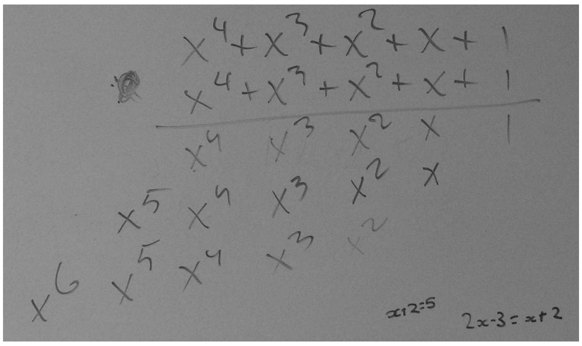

DW was intrigued and made a couple of experiments (and it appeared from his behaviour that he was using mostly mental arithmetic, writing down the result, term by term, with pauses):



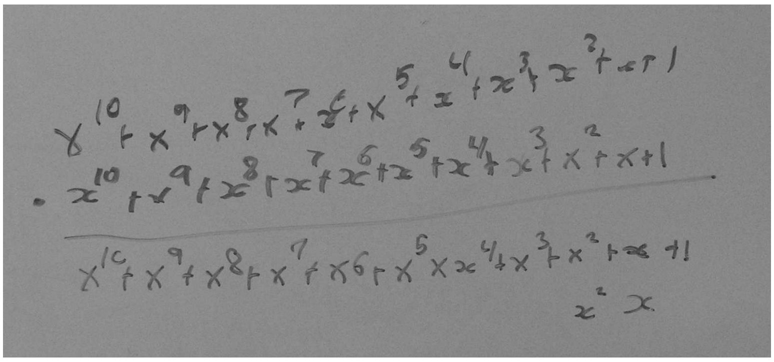

and said with obvious enthusiasm: "Yes, it is the same pattern!"

"Wonderful"–answered I–"let us see why this is happening. I'll give you a hint: multiplication of polynomials can be written as column multiplication", and started to write:



DW did not let me finish, grabbed the marker from my hand and insisted on doing it himself:



He stopped after he barely started the second line and said very firmly: "Yes, it is like with numbers".

"Well"–said I–"but will the pattern break down or will continue forever?"

That was the first time when DW fell in deep thought (and I was a bit uncomfortable about the degree of his concentration and retraction from the real world). This was also the first time when his response was not instant–perhaps, whole 20 seconds passed in silence. Then he suddenly smiled happily and answered: "No, it will not break down!"

"Why?"–inquired I.

"Because when you add polynomials, the coefficients just add up, there are no carries."

At that point I decided to stop the session on the pretext that it was late and the boy was perhaps tired, but, to round up the discussion, made a general comment:

"You know, in mathematics polynomials are sometimes used to explain what is happening with numbers".

The last word, however, belonged to DW:

"Yes, 10 is $x$."

## 11. DW: a discussion

DW is a classical example of what is usually called a "mathematically able child". He mastered, more or less on his own, some mathematical routines—multiplication of decimals and polynomials—which are normally taught to children at much later age. He also showed instinctive interest in detecting beautiful patterns in behaviour of numbers, and, which is even more important, in limits of applicability of patterns, in their breaking points.

DW understands what generalisation is and, moreover, loves making quick, I would say recklessly quick generalisations. Krutetskii [**16**] lists this trait among characteristic traits of "mathematically able" children: very frequently, they are children, who, after solving just one problem, already know how to solve *any* problem of the same type.

But let us return to the principal theme of the present paper: hidden structures of elementary mathematics. In our conversation, DW was shown—I emphasise, for the first time in his life—a beautiful but hidden connection between decimals and polynomials—*and was able to see it*!

In our little exercise, DW advanced (a tiny step) in *conceptual* understanding of mathematics: he had seen an example of how one mathematical structure (polynomials) may hide inside another mathematical structure (decimals).

My final comment is that although DW made a small, but important step towards deeper understanding of mathematics, this step is not necessarily visible in the standard mathematics education framework. It is unlikely that a school assignment will detect him making this small step. Procedurally, in this small exercise DW learned next to nothing—he multiplied numbers and polynomials before, he will multiply them with the same speed after.

One should not think, however, that the "procedural" aspect of mathematics is of no importance. DW's ability to do this tiny bit of "conceptual" mathematics would be impossible without him mastering the standard routines (in this case, column multiplication of decimals and addition and multiplication of polynomials).

## 12. Carry, Cinderella of arithmetic

The deceptive simplicity of elementary school arithmetic is especially transparent when we take a closer look at *carries* in the addition of decimals.

**12.1. Cohomology.** In Molièr's *Le Bourgeois Gentilhomme*, Monsieur Jourdain was surprised to learn that he had been speaking prose all his life. I was recently reminded that, starting from my elementary school and then all my life, I was calculating 2-cocycles.

Indeed, a carry in elementary arithmetic, a digit that is transferred from one column of digits to another column of more significant digits during addition of two decimals, is defined by the rule

$$c(a,b) = \begin{cases} 1 & \text{if } a+b > 9 \\ 0 & \text{otherwise} \end{cases}.$$

One can easily check that this is a 2-cocycle from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}$ and is responsible for the extension of additive groups

$$0 \longrightarrow 10\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0.$$

DW discovered (without knowing the words '2-cocycle' and 'cohomology') that carry is doing what cocycles frequently do: they are responsible for break of symmetry.

**12.2. A few formal definitions.** Let $G$ be a group and $A$ an abelian group with an action of $G$ on $A$:

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ (g,a) & \mapsto & g \cdot a. \end{array}$$

A 2-*cocycle* is a map

$$f : G \times G \longrightarrow A$$

such that

$$g \cdot f(h,k) + f(g,hk) = f(gh,k) + f(g,h)$$

Let $E$ be an extension of $A$ by $G$,

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1,$$

$S$ a system of coset representative of $A$ in $E$,

$$s : G \longrightarrow E$$

a coset map.

Then $f : G \times G \to A$ defined by

$$s(gh) = f(g,h)s(g)s(h)$$

is a 2-cocycle. It measures the extent to which the collection of coset representatives fails to be closed under multiplication.

A *2-coboundary* for the action of $G$ on $A$ is a function

$$f : G \times G \to A$$

such that there exists a function

$$\phi : G \to A$$

such that:

$$f = (g,h) \mapsto g \cdot \phi(h) - \phi(gh) + \phi(g)$$

Two 2-cocycles for the same extension $E$ differ by a 2-coboundary. Therefore the exensions are described by the second cohomology group

$$H^2(G,A) = Z^2(G,A)/B^2(G,A),$$

where $Z^2(G,A)$ is the group of 2-cocycles with respect to natural pointwise addition and $B^2(G,A)$ is the group of coboundaries.

**12.3. Limits and series.** Carry has another interesting property: it contains a seed of a concept of limit leading to immensely rich $p$-adic analysis.

A few words on limits are due. A rare topic in undergraduate education generates more controversy than the classical $\epsilon$-$\delta$ approach to limits and continuity in the real domain. I quote Raphael Núnez [**21**]:

> Formal definitions and axioms in mathematics are themselves created by human ideas [...] and they only capture very limited aspects of the richness of mathematical ideas. Moreover, definitions and axioms often neither formalize nor generalize human everyday concepts. A clear example is provided by the modern definitions of limits and continuity, which were coined after the work by Cauchy, Weierstrass, Dedekind, and others in the 19th century. These definitions are at odds with the inferential organization of natural continuity provided by cognitive mechanisms such as fictive and metaphorical motion. Anyone who has taught calculus to new students can tell how counter-intuitive and hard to understand the epsilon-delta definitions of limits and continuity are (and this is an extremely well-documented fact in the mathematics education literature). The reason is (cognitively) simple. Static epsilon-delta formalisms neither formalize nor generalize the rich human dynamic concepts underlying continuity and the "approaching" the location.

This thesis is fully developed in a book by Lakoff and Núnez [**17**] and is very representative of a certain school of thought in mathematics education largely informed by neurophysiological research.

In this context, it is interesting to analyse how $p$-adic analysis arises from the purely algebraic concept of carry and completely avoids all alleged psychological traps which imperil the study of real analysis.

Very frequently, when we deal with a mathematical object and wish to modify it and make it "infinite" in some sense, we have several different ways for doing so. For example, usual decimal numbers can be extended to infinite decimal expansions to the *right*:

$$\pi = 3.1415926\ldots$$

and to the *left*:

$$\ldots 987654321$$

In the second case, the operations of multiplication and addition on infinite to the left decimals (called 10-*adic integers*) are defined in the usual way, with the excess carried to the next position on the left. Carries march on and on, uninterruptedly, and this steadiness of their pace is the psychological basis of a very intuitive concept of limit.

10-adics are not frequently used in mathematics, but $p$-adic integers for prime values of the base $p$, defined in a similar way by expanding integers written to base $p$ to the left, are quite useful and very popular.

10-adic integers are not so good as $p$-adic for prime $p$ because they contain zero divisors, non-zero numbers $x$ and $y$ such that $xy = 0$. The following elementary example was provided by Hovik Khudaverdyan and Gábor Megyesi and nicely illustrates the concept of 10-adic limit. If you look at the sequence of iterated squares

$$5,\ 5^2 = 25,\ 25^2 = 625,\ 625^2 = 390625,\ 390625^2 = 152587890625\ldots$$

you notice that consecutive numbers have in common an increasingly long sequences of the rightmost digits, that is,

$$5^{2^{n+1}} \equiv 5^{2^n} \mod 10^n,$$

the fact which could be easily proven by induction. This freezing of rightmost digits means exactly that the sequence converges to a 10-adic integer

$$x = \ldots 92256259918212890625.$$

One can easily see that $x$ has the property that $x^2 = x$ and hence $x(x-1) = 0$. Therefore $x$ and $x - 1$ are desired zero divisors.

It can be shown that zero divisors appear in the ring of 10-adic integers because 10 is not a prime number. An exercise for the reader: prove that the ring of 2-adic integers has no zero divisors.

Properties of 10-adic and $p$-adic integers are quite different from that of real numbers: to give one example, you cannot order 10-adic integers in a way compatible with addition and multiplication (so that the usual rules of manipulating inequalities would hold). This can be seen already from one of the simplest instances of addition:

$$\ldots 99999 + 1 = \ldots 00000 = 0.$$

For lack of space, I will mention only one other property of $p$-adic integers made self-explanatory by application of simplest rules of operating with carries.

**12.4. Euler's sum.** Notice that a paradoxical summation of the infinite series

$$1 + 2 + 4 + \cdots = -1,$$

due to Euler, makes sense and is completely correct in the domain of 2-adic integers written by base 2 expansions. Indeed, it becomes an easy-to-check arithmetic calculation:

$$\begin{aligned} 1 + 2 + 4 + 8 + \cdots &= 1 + 10 + 100 + 1000 + \cdots \\ &= \cdots 111111111 \end{aligned}$$

But

$$\begin{array}{rr} & \cdots 111111111 \\ + & 1 \\ = & \cdots 000000000 \end{array}$$

Hence in 2-adic arithmetic

$$1 + 2 + 4 + 8 + \cdots = -1.$$

It was worth noting that Euler was most likely to know binary numbers. A first clear description of them was published in 1703 by Leibnitz in his paper *Explication* *de l'Arithétique Binaire* [**19**] in modern notation, se Figure 11:

$$\begin{aligned} 0 &= 0 \\ 1 &= 1 \\ 2 &= 10 \\ 3 &= 11 \\ 4 &= 100 \\ 5 &= 101 \\ 6 &= 110 \\ 7 &= 111 \\ &\vdots \\ 2^n &= 10\cdots 0 \quad (\ n \text{ zeroes}) \end{aligned}$$

**Exercises.**

EXERCISE 12.1. (Gardiner [**12**]) The list of numbers

$$49, \quad 4489, \quad 444889, \quad 44448889, \quad 4444488889, \ldots$$

goes on for ever. What is the next number on the list? The first number 49 is a perfect square. Is the second number 4489 a perfect square? Is the tenth number on the list a perfect square? Which other numbers in the list are perfect squares?

EXERCISE 12.2. Prove that a 10-adic integer has an inverse if and only if its last digit is 1, 3, 7 or 9.

Hint. If the last digit of $x$ is 1, we can write it as $x = 1 + 10y$ and

$$(1 + 10y)^{-1} = 1 - 10y + 100y^2 - 100y^3 + \cdots,$$

with the expression on the right making sense because in the infinite sum in every position we sum up only finitely many digits.

EXERCISE 12.3. Prove that a 2-adic integer has an inverse if and only if its last digit is 1.

EXERCISE 12.4. Prove that every 2-adic integer can be written in the form $2^k \cdot x$, where $k$ is an (ordinary) non-negative integer and $x$ is an invertible 2-adic integer.

EXERCISE 12.5. Prove that the ring $\mathbb{Z}_2$ of 2-adic integers is a domain, that is, it has no zero divisors.

EXERCISE 12.6.

EXERCISE 12.7. Study the arithmetic properties of the ring $\mathbb{Z}[[x]]$ of formal power series over integers; here, formal power series are expressions

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

with all coefficients $a_i \in \mathbb{Z}$, with usual operations of addition and multiplication.

- Is $\mathbb{Z}[[x]]$ a domain? (*Domain* is a commutative ring which has no zero divisors.)
- Which elements of $\mathbb{Z}[[x]]$ are invertible?

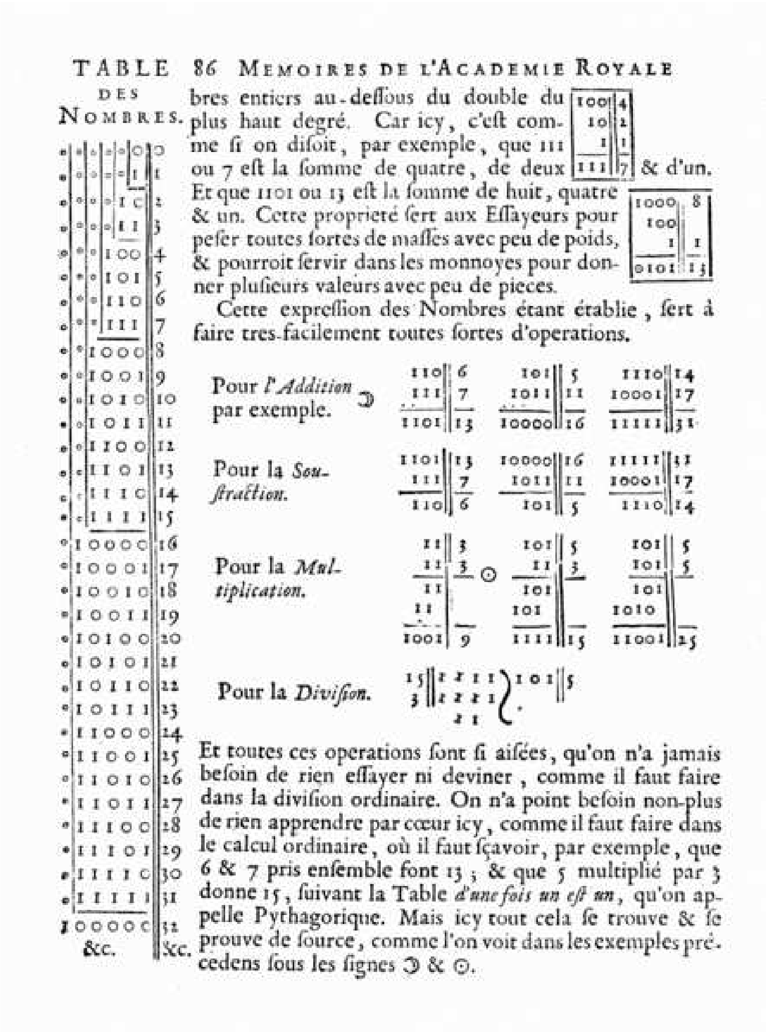


TABLE DES NOMBRES. 86 MEMOIRES DE L'ACADEMIE ROYALE

bres entiers au-deſſous du double du plus haut degré. Car icy, c'eſt comme ſi on diſoit, par exemple, que 111 ou 7 eſt la ſomme de quatre, de deux & d'un. Et que 1101 ou 13 eſt la ſomme de huit, quatre & un. Cette proprieté ſert aux Eſſayeurs pour peſer toutes ſortes de maſſes avec peu de poids, & pourroit ſervir dans les monnoyes pour donner pluſieurs valeurs avec peu de pieces.

Cette expreſſion des Nombres étant établie, ſert à faire tres-facilement toutes ſortes d'operations.

Pour l'*Addition* par exemple. ☽

Pour la *Souſtraction.*

Pour la *Multiplication.* ⊙

Pour la *Diviſion.*

Et toutes ces operations ſont ſi aiſées, qu'on n'a jamais beſoin de rien eſſayer ni deviner, comme il faut faire dans la diviſion ordinaire. On n'a point beſoin non-plus de rien apprendre par cœur icy, comme il faut faire dans le calcul ordinaire, où il faut ſçavoir, par exemple, que 6 & 7 pris enſemble font 13; & que 5 multiplié par 3 donne 15, ſuivant la Table *d'une fois un eſt un*, qu'on appelle Pythagorique. Mais icy tout cela ſe trouve & ſe prouve de ſource, comme l'on voit dans les exemples precedens ſous les ſignes ☽ & ⊙.

FIGURE 11. A page from Leibnitz's paper about binary system [**19**], 1703.

- Take a page from DW's book and find in $\mathbb{Z}[[x]]$ the inverse of

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

- Does $\mathbb{Z}[[x]]$ have unique factorisation?

See Birmajer and Gill [**4**] for answers.

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