Metamathematics of Elementary Mathematics

Lecture 1

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Introduction

Toutes les grandes personnes ont d'abord été des enfants
(Mais peu d'entre elles s'en souviennent.)
Antoine de Saint-Exupéry, *Le Petit Prince.*

These lectures is an attempt to gain a better understanding of the specific nature of mathematical practice by looking at mathematics from a new and somewhat unusual point of view. I propose to systematically record and analyse *mathematically* logical difficulties experienced—and occasionally overcome—by children in their early learning of mathematics. This explains the title of the course: according to a widely accepted understanding of this term, *metamathematics* is mathematics applied to study of mathematics. In my case I apply mathematics for the sake of better understanding of the practice of elementary school mathematics.

Quite naturally, the real life material in my research is limited to analysis of recollections of my fellow mathematicians about their childhood. Unfortunately, only mathematicians posses an adequate language which allows them to describe in some depths their personal experiences of learning mathematics. So far my approach is justified only by warm welcome it found in some of my mathematician friends, and I am most grateful to them for their support.

I am a mathematician and restrict myself to clearly describing hidden structures of elementary mathematics which may intrigue and—like shadows in the night—sometimes scare an inquisitive child. I hope that my notes could be useful to specialists in mathematical education and in psychology of education. But I refrain from making any recommendations on mathematics teaching. For me, the primary aim of my project is to understand the nature of hardcore mainstream "research" mathematics.

The emphasis on child's experiences makes my programme akin to linguistic and cognitive science. However, when a linguist studies formation of speech in a child, he studies language, not the structure of linguistic as a scientific discipline. When I propose to study formation of mathematical concepts in a child, I wish to get insights into interplay of mathematical structures in *mathematics.* Mathematics has an astonishing power of reflection, and a self-referential study of mathematics by mathematical means plays an increasingly important role within mathematical

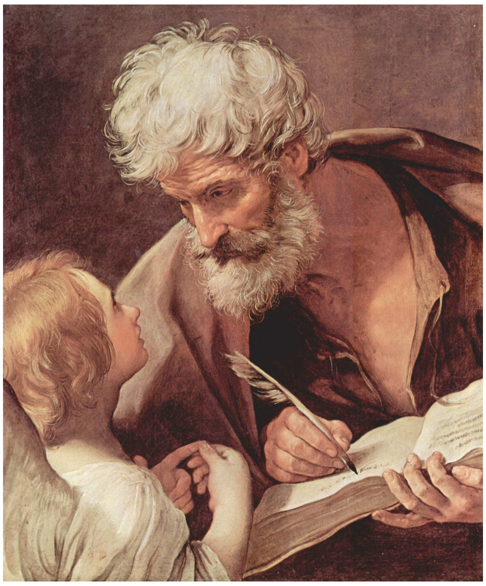

Figure 1. *L'Evangelista Matteo e l'Angelo.* Guido Reni, 1630–1640. Pinacoteca Vaticana. Source: *Wikipedia Commons*. Public domain. Guido Reni was one of the first artists in history of visual arts who paid attention to psychology of children. Notice how the little angel counts on his fingers the points he is sent to communicate to St. Matthew.

culture. I simply suggest to make a step further (or aside, or back in life) and take a look back in time, in one's child years.

The lectures are also an attempt to trigger the chain of memories in my listeners: every, even most minute, recollection of difficulties and paradoxes of their early mathematical experiences are most welcome. Please write to me at

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Contents

Part 1. Dividing apples between people

1. "Named" numbers

1.1. Dividing apples between people. I take the liberty to tell a story from my own life[1]; I believe it is relevant for the principal theme of the paper.

When, as a child, I was told by my teacher that I had to be careful with *"named"* numbers and not to *add* apples and people, I remember asking her why in that case we can *divide* apples by people:

$$10\,\text{apples} : 5\,\text{people} = 2\,\text{apples}. \tag{1}$$

Even worse: when we distribute 10 apples giving 2 apples to a person, we have

$$10\,\text{apples} : 2\,\text{apples} = 5\,\text{people} \tag{2}$$

Where do "people" on the right hand side of the equation come from? Why do "people" appear and not, say, "kids"? There were no "people" on the left hand side of the operation! How do numbers on the left hand side know the name of the number on the right hand side?

There were much deeper reasons for my discomfort. I had no bad feelings about dividing 10 apples among 5 people, but I somehow felt that the problem of deciding how many people would get apples if each was given 2 apples from the total of 10, was completely different. (My childhood experience is confirmed by experimental studies, see Bryant and Squire [**6**].)

In the first problem you have a fixed data set: 10 apples and 5 people, and you can easily visualize giving apples to the people, in rounds, one apple to a person at a time, until no apples were left. But an attempt to visualize the second problem in a similar way, as an orderly distribution of apples to a queue of people, two apples to each person, necessitated dealing with a potentially unlimited number of recipients. In horror I saw an endless line of poor wretches, each stretching out his hand, begging for his two apples. This was visualization gone astray. I was not in control of the queue! But reciting numbers, like chants, while counting *pairs*

[1]Call me AVB; I am male, have a PhD in Mathematics, teach mathematics at a research-led university.

FIGURE 2. The First Law of Arithmetic: you do not add fruit and people. Giuseppe Arcimboldo, *Autumn*. 1573. Musée du Louvre, Paris. Source: *Wikipedia Commons*. Public domain.

of apples, had a soothing, comforting influence on me and restored my shattered confidence in arithmetic.

I did not get a satisfactory answer from my teacher and only much later did I realize that the correct naming of the numbers should be

$$10\,\text{apples} \;:\; 5\,\text{people} = 2\,\frac{\text{apples}}{\text{people}}, \qquad 10\,\text{apples} \;:\; 2\,\frac{\text{apples}}{\text{people}} = 5\,\text{people}. \tag{3}$$

It is a commonplace wisdom that the development of mathematical skills in a student goes alongside the gradual expansion of the realm of numbers with which he or she works, from natural numbers to integers, then to rational, real, complex numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

What is missing from this natural hierarchy is that already at the level of elementary school arithmetic children are working in a much more sophisticated structure, a graded ring

$$\mathbb{Q}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}].$$

of Laurent polynomials in n variables over $\mathbb{Q}$, where symbols $x_1, \ldots, x_n$ stand for the names of objects involved in the calculation: apples, persons, etc. This explains why educational psychologists confidently claim that the operations (1) and (2) have little in common [**6**]—indeed, operation (2) involves an operand "apple/people" of a much more complex nature than basic "apples" and "people" in operation (1): "apple/people" could appear only as a result of some previous division.

Usually, only Laurent monomials are interpreted as having physical (or real life) meaning. But the addition of heterogeneous quantities still makes sense and is done componentwise: if you have a lunch bag with (2 apples + 1 orange), and another bag, with (1 apple + 1 orange), together they make

$$(2 \text{ apples } + 1 \text{ orange}) + (1 \text{ apple } + 1 \text{ orange}) = (3 \text{ apples } + 2 \text{ oranges}).$$

Notice that this gives a very intuitive and straightforward approach to vectors.[2]

Of course, there is no need to teach Laurent polynomials to kids; but it would not harm to teach them to teachers. I have an ally in François Viéte who in 1591 wrote in his *Introduction to the Analytic Art* [**27**] that

> *If one magnitude is divided by another, [the quotient] is heterogeneous to the former ... Much of the fogginess and obscurity of the old analysts is due to their not paying attention to these [rules].*

1.2. Dimensions. Physicists love to work in the Laurent polynomial ring

$$\mathbb{R}[\text{length}^{\pm 1}, \text{time}^{\pm 1}, \text{mass}^{\pm 1}]$$

because they love to measure all physical quantities in combinations (called "dimensions") of the three basic units: for length, time and mass. But then even this ring becomes too small since physicists have to use fractional powers of basic units. For example, velocity has dimensions length/time, while electric charge can be meaningfully treated as having dimensions

$$\frac{\text{mass}^{1/2}\text{length}^{3/2}}{\text{time}}.$$

Indeed if we choose our units in such a way that the permittivity ϵ_0 of free space is dimensionless, then from Coulomb's law

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

applied to two equal charges $q_1 = q_2 = q$, we see that q^2/r^2 has the dimensions of force.

It pays to be attentive to the dimensions of quantities involved in a physical formula: the balance of names of units (*dimensions*) on the left and right hand sides may suggest the shape of the formula. Such *dimensional analysis* quickly leads to immensely deep results, like, for example, Kolmogorov's celebrated "5/3 Law" for the energy spectrum of turbulence.

[2] By the way, this "lunch bag" approach to vectors allows a natural introduction of duality and tensors: the total cost of a purchase of amounts g_1, g_2, g_3 of some goods at prices p^1, p^2, p^3 is a "scalar product"-type expression $\sum g_i p^i$. We see that the quantities g_i and p_i could be of completely different nature. The standard treatment of scalar (dot) product in undergraduate linear algebra usually conceals the fact that dot product is a manifestation of duality of vector spaces, creating immense difficulties in the subsequent study of tensor algebra.

Exercises.

Exercise 1.1. To scare the reader into acceptance of the intrinsic difficulty of division, I refer to paper *Division by three* [**10**] by Peter Doyle and John Conway. I quote their abstract:

> We prove without appeal to the Axiom of Choice that for any sets A and B, if there is a one-to-one correspondence between $3 \times A$ and $3 \times B$ then there is a one-to-one correspondence between A and B. The first such proof, due to Lindenbaum, was announced by Lindenbaum and Tarski in 1926, and subsequently 'lost'; Tarski published an alternative proof in 1949.

Here, of course, 3 is a set of 3 elements, say, $\{0, 1, 2\}$. An exercise for the reader: prove this in a naive set theory *with* the Axiom of Choice.

Exercise 1.2. Theoretical physicists occasionally love to use a system of measurements based on fundamental units:

- speed of light $c = 299,792,458$ meters per second,
- gravitational constant $G = (6.67428 \pm 0.00067) \times 10^{-11}$ $\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ and
- Planck' constant $h = 6.62606896 \times 10^{-34}$ $\mathrm{m}^2\mathrm{kg} \cdot \mathrm{s}^{-1}$.

Express the more common physical units: meter, kilogram, second in terms of c, G, h.

2. The triumph of "named numbers": Kolmogorov's "5/3" Law

To demonstrate the power of "named numbers", I reproduce what remains the most striking and beautiful example of dimensional analysis in mathematics. The deduction of Kolmogorov' seminal "5/3" law for the energy distribution in the turbulent fluid [**15**] is so simple that it can be done in a few lines. I was lucky to study at a good secondary school where my physics teacher (Anatoly Mikhailovich Trubachov, to whom I express my eternal gratitude) derived the "5/3" law in one of his improvised lectures. In my exposition, I borrow some details from Arnold [**2**] and Ball [**3**] (where I have also picked the idea of using a woodcut by Katsushika Hokusai, Figure 3, as an illustration).

2.1. Turbulent flows: basic setup. The turbulent flow of a liquid consists of vortices; the flow in every vortex is made of smaller vortices, all the way down the scale to the point when the viscosity of the fluid turns the kinetic energy of motion into heat (Figure 3). If there is no influx of energy (like the wind whipping up a storm in Hokusai's woodcut), the energy of the motion will eventually dissipate and the water will stand still. So, assume that we have a balanced energy flow, the storm is already at full strength and stays that way. The motion of a liquid is made of waves of different lengths; Kolmogorov asked the question, what is the share of energy carried by waves of a particular length?

Here it is a somewhat simplified description of his analysis. We start by making a list of the quantities involved and their dimensions. First, we have the *energy flow* (let me recall, in our setup it is the same as the dissipation of energy). The dimension of energy is

$$\frac{\text{mass} \cdot \text{length}^2}{\text{time}^2}$$

FIGURE 3. Multiple scales in the motion of a fluid, from a woodcut by Katsushika Hokusai *The Great Wave off Kanagawa* (from the series *Thirty-six Views of Mount Fuji*, 1823–29). This image is much beloved by chaos scientists. Source: *Wikipedia Commons.* Public domain.

(remember the formula $K = mv^2/2$ for the kinetic energy of a moving material point). It will be convenient to make all calculations *per unit of mass.* Then the energy flow ϵ has dimension

$$\frac{\text{energy}}{\text{mass} \cdot \text{time}} = \frac{\text{length}^2}{\text{time}^3}$$

For counting waves, it is convenient to use the *wave number*, that is, the number of waves fitting into the unit of length. Therefore the wave number k has dimension

$$\frac{1}{\text{length}}.$$

Finally, the *energy spectrum* $E(k)$ is the quantity such that, given the interval $\Delta k = k_1 - k_2$ between the two wave numbers, the energy (per unit of mass) carried by waves in this interval should be approximately equal to $E(k_1)\Delta k$. Hence the dimension of E is

$$\frac{\text{energy}}{\text{mass} \cdot \text{wavenumber}} = \frac{\text{length}^3}{\text{time}^2}.$$

2.2. Subtler analysis. To make the next crucial calculations, Kolmogorov made the major assumption that amounted to saying that[3]

> The way bigger vortices are made from smaller ones is the same throughout the range of wave numbers, from the biggest vortices

[3] This formulation is a bit cruder than most experts would accept; I borrow it from Arnold [**2**].

Figure 4. Andrei Nikolaevich Kolmogorov, 1903–1987

(say, like a cyclone covering the whole continent) to a smaller one
(like a whirl of dust on a street corner).

Then we can assume that the energy spectrum E, the energy flow ϵ and the wave number k are linked by an equation which does not involve anything else. Since the three quantities involved have completely different dimensions, we can combine them only by means of an equation of the form

$$E(k) \approx C\epsilon^x \cdot k^y.$$

Here C is a constant; since the equation should remain the same for small scale and for global scale events, the shape of the equation should not depend on the choice of units of measurements, hence C should be dimensionless.

Let us now check how the equation looks in terms of dimensions:

$$\frac{\text{length}^3}{\text{time}^2} = \left(\frac{\text{length}^2}{\text{time}^3}\right)^x \cdot \left(\frac{1}{\text{length}}\right)^y.$$

After equating lengths with lengths and times with times, we have

$$\begin{aligned} \text{length}^3 &= \text{length}^{2x} \cdot \text{length}^{-y} \\ \text{time}^2 &= \text{time}^{3x}, \end{aligned}$$

which leads to a system of two simultaneous linear equations in x and y,

$$\begin{aligned} 3 &= 2x - y \\ 2 &= 3x \end{aligned}$$

This can be solved with ease and gives us

$$x = \frac{2}{3} \quad \text{and} \quad y = -\frac{5}{3}.$$

Therefore we come to *Kolmogorov's "5/3" Law*:

$$E(k) \approx C\epsilon^{2/3}k^{-5/3}.$$

We have done it!

It is claimed in the specialist literature that the dimensionless constant C can be determined from experiments and happens to be pretty close to 1.

2.3. Discussion. The status of this celebrated result is quite remarkable. In the words of an expert on turbulence, Alexander Chorin [**7**],

> Nothing illustrates better the way in which turbulence is suspended between ignorance and light than the Kolmogorov theory of turbulence, which is both the cornerstone of what we know and a mystery that has not been fathomed.
>
> The same spectrum [...] appears in the sun, in the oceans, and in manmade machinery. The 5/3 law is well verified experimentally and, by suggesting that not all scales must be computed anew in each problem, opens the door to practical modelling.

Arnold [**2**] reminds us that the main premises of Kolmogorov's argument remain unproven—after more than 60 years! Even worse, Chorin points to the rather disturbing fact that

> Kolmogorov's spectrum often appears in problems where his assumptions clearly fail. [...] The 5/3 law can now be derived in many ways, often under assumptions that are antithetical to Kolmogorov's. Turbulence theory finds itself in the odd situation of having to build on its main result while still struggling to understand it.

3. History of dimensional analysis

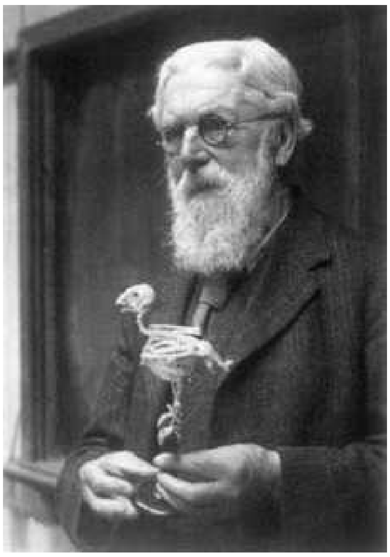

FIGURE 5. D'Arcy Wentworth Thompson, 1860–1948

3.1. Froude's Law of Steamship Comparisons. It would be interesting to have an account of the history of dimensional analysis. It can be traced back at least to *Froude's Law of Steamship Comparisons* used to great effect in D'Arcy Thompson's book *On Growth and Form* [**25**, p. 24] for the analysis of speeds of animals:

> the maximal speed of similarly designed steamships is proportional to the square root of their length.

William Froude (1810–1879) was the first to formulate reliable laws for the resistance that water offers to ships and for predicting their stability. In this section, we give a deduction of law adapted from [**25**, p. 24]. But first we have to discuss some difficulties of mathematical modelling of physical phenomena and limitations of dimensional analysis.

3.2. Difficulty of making physical models. We need to understand first how the drag F (force of resistance) offered by water to a ship depends on the speed of the ship. Surprisingly, it is is easier to do for high speeds than for lower ones. At high speeds we can assume that the drag is offered by water being violently thrown away off the course of the ship, ignoring a finer picture of what is happening with the water. Obviously, the drag F should depend on the crossection area S of the ship, its speed V and density of water ρ: heavier the water, harder is to kick it off. Therefore we are looking for an equation in the form

$$F = c\rho^x S^y V^z$$

with a dimensionless coefficient c. Substituting dimensions, we have

$$\frac{\text{mass}\cdot\text{length}}{\text{time}^2} = \left(\frac{\text{mass}}{\text{length}^3}\right)^x \cdot \left(\text{length}^2\right)^y \cdot \left(\frac{\text{length}}{\text{time}}\right)^z,$$

and immediately see that $z = 2$, $x = y = 1$, and the so-called *Formula of Quadratic Drag* takes the form

$$F = c\rho S V^2.$$

According to Wikipedia, this equation is attributed to Lord Rayleigh.

At lower speeds, water smoothly flows around the hull of the ship, and a much more delicate analysis is needed. It has been done by Sir George Stokes[4] (1819–1903) for the special case of small spherical objects moving slowly through a viscous fluid. It is

$$F = -6\pi\eta r V,$$

where r is the effective radius of the object, η is the viscosity of the fluid.

In short, for things like bacteria the drag is proportional to the length and to velocity; we shall denote that symbolically as

$$F \sim LV.$$

Notice that the proportionality coefficient is no longer dimensionless.

Alas, for ships Stokes' beautiful mathematical deduction does not work.

[4] Stokes is likely to be known to the reader as the author of the Stokes formula for surface integrals.

Figure 6. William Froude, 1810–1879

3.3. Deduction of Froude's Law. Instead of Raleigh's and Stokes' drag formulae, we have to use the property (and we shall treat it simply as an experimental fact, as Froude did after a number of experiments) that, at small speeds, the drag F is proportional to the area of cross section of the ship (and hence to the square L^2 of its typical linear size L) and to the velocity V of the ship. We shall write it symbolically as

$$F \sim L^2V$$

and call *Froude's drag.*

To sustain constant speed, the ship has to produce power FV. Now we have to recall that we are talking about *steamships*, powered by an engine inside, this engine working on steam produced by burning coal, this coal being carried by the ship, etc. Therefore it is reasonable to assume that the power produced by the ship is proportional to its volume, and therefore

$$FV \sim L^3.$$

Combining the two proportions together, we have

$$L^2V^2 \sim L^3$$

and therefore

$$V \sim L^{1/2}.$$

D'Arcy Thompson applied the same derivation to fish because of the validity of the principal assumption: the energy is produced internally.

Exercises.

Exercise 3.1. How would Froude's Law look for solar powered ships? (Use Froude's drag.)

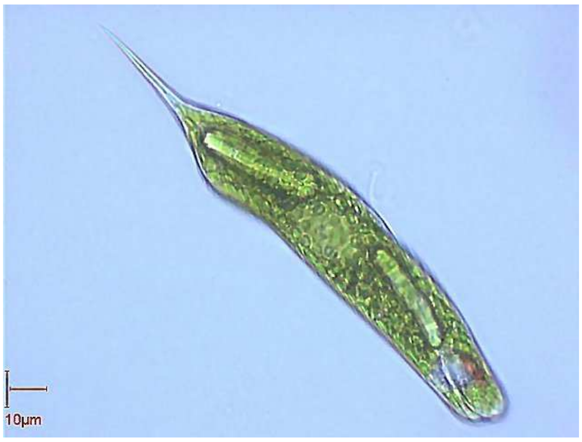

FIGURE 7. *Euglena*, one of many single cell solar powered living organisms able to move in water, by waggling it tale (*flagellum*). © BIODIDAC, http://biodidac.bio.uottawa.ca/info/regles.htm

EXERCISE 3.2. Try to relate the previous exercise to the fact that all known living organisms capable of photosynthesis (and thus able to use solar power directly) while also able to move in water consist of a single cell, see Figure 7.

EXERCISE 3.3. Compare maximal speeds of photosynthesising (that is, solar powered) and phagocytosising (that is, fuel powered) single cell organisms. Use Stokes' drag.

EXERCISE 3.4. Why does a mouse have (relatively) slimmer body build than an elephant?

EXERCISE 3.5. Prove the following corollary of Froude's Law:

> The *relative* speed of a fish (that is, speed measured in numbers of its lengths covered by the fish per unit of time) is *inverse proportional* to the square root of its length.

This explains a well-known phenomenon: little fish in a stream appear to be *very* quick.

EXERCISE 3.6. Estimate, who is relatively faster: an ant or a racehorse?

EXERCISE 3.7. Deduce a version of Froude's Law for high speeds and quadratic drag.

EXERCISE 3.8. Analyse the following quote from Greenhill [**13**, p. 233]:

> Thus treating a man's leg as swinging like a pendulum through a fixed definite angle, the length of a pace is proportional to his height, but the number of paces a minute is inversely as the square root of the height; so that his pace of walking and getting over the ground will vary as the square root of his height, as in Froude's Law again.

Using dimensional analysis, prove:

FIGURE 8. Walking on stilts increases speed (for Exercise 3.8).

- the number of swings of a pendulum a minute is inversely as the square root of the length of the pendulum;

and use this fact to derive Greenhill's conclusion:

- his pace of walking and getting over the ground will vary as the square root of his height (or of length of legs, see Figure 8).

Investigate further:

- how do the frequency of a pendulum's swings and the pace of a man's walking depend on the acceleration of gravity? (Read more on that in [**26**].)

EXERCISE 3.9. Building on ideas from Exercise 3.4, develop a method for estimation of a maximal possible height of a tree of given species.

The problem is of serious practical value. To explain it to an Englishman, you have to mention just one word: *Leylandii.* For a foreigner, *Wikipedia* provides an explanation:

> The Leyland Cypress, *X Cupressocyparis leylandii*, is often referred to as just Leylandii. It is a fast-growing evergreen tree much used in horticulture, primarily for hedges and screens.
>
> The Leyland Cypress is a hybrid between the Monterey Cypress, *Cupressus macrocarpa*, and the Nootka Cypress, *Cupressus nootkatensis.* The hybrid has arisen on nearly 20 separate occasions, always by open pollination. [...]
>
> Leyland Cypresses are commonly planted in gardens to provide a quick boundary or shelter hedge. However, their rapid growth (up to a metre per year), heavy shade and great potential height (often over 20 m tall in garden conditions, they can reach at least 35 m) make them a problem. In Britain they have been the source of a number of high profile disputes between neighbours, even leading to violence (and in one recent case, murder), because of their capacity to cut out light.

Figure 9. *X Cupressocyparis leylandii*, 35 meters and still growing.

The problem is that no-one knows the maximal height of some of the latest hybrids—all known specimens continue to grow...

I have to admit that I do not know an answer to this exercise and invite my listeners treat it as an open problem.

FIGURE 14. Guido Reni. A fragment of *Purification of the Virgin*, c. 1635–1640. Musée du Louvre. Source: *Wikipedia Commons*. Public domain.

Acknowledgements

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This text would not appear without kind invitations to give a talk at "Is Mathematics Special" conference in Vienna in May 2008 and to give a lecture course "Elementary mathematics from the point of view of "higher" mathematics" at the Nesin Mathematics Village in Şirince, Turkey, in July 2008.

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